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Properties Preservation of Expansion of Models of NIP Theories

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CONTENTS

NORMATIVE REFERENCES

There are used next references for standards in the present thesis:

SOSE RK 5.04.034-2011. State obligatory standard of education. Postgraduate education. Doctoral studies.

State standard 7.32-2001. Report on scientific-research work. Structure and rules of presentation.

State Standard 7.1-2003. Bibliographic record. Bibliographic description. General requirements and rules.

DEFINITIONS

There are used next designators with the corresponding definitions in the dissertation. At the beginning we give the main model-theoretic determinations and designators. [1-7]. We will introduce more complicated notions in further sections.

A **signature** or a **language** Σ (or L) consists of the following symbols:
1) functional symbols f_i
2) relational symbols R_i
3) constant symbols c_i

An L-structures $N = \langle N; f_i, R_i, c_i \rangle_{i\in\mathbb{N}}$ is defined as follows: an **universe** N of a structure with f^N: $N^{f_i} \to N$, $R_i \subseteq N$ and c_i $from$ N.

We say that A is *convex* in N for subset A of a linearly ordered structure N if it follows $a < c < b$ where $c \in A$ for any $a, b \in A$ such that $a < b$.

Let $\mathcal{M}, \mathcal{N}$ be structures of a signature Σ. We say that $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ (denoted by $\mathcal{M} \subset \mathcal{N}$) if for any quantifier free formula $\psi(z_1, \ldots, z_n)$ and for any $c_1, \ldots, c_n$ $from$ M the next holds:

$$\mathcal{M} \vDash \psi(c_1, \ldots, c_n) \Leftrightarrow \mathcal{N} \vDash \psi(c_1, \ldots, c_n).$$

Suppose $\mathcal{M}$, $\mathcal{N}$ are structures of a signature Σ. $\mathcal{M}$ is called an *elementary submodel* of $\mathcal{N}$(notation is $\mathcal{M}°\mathcal{N}$) if for every formula $\psi(z_1, \ldots, z_n)$ and for all $c_1, \ldots, c_n$ $from$ M the next holds:

$$\mathcal{M} \vDash \psi(c_1, \ldots, c_n) \Leftrightarrow \mathcal{N} \vDash \psi(c_1, \ldots, c_n).$$

We say that a formula $U(x, y)$ is *convex to the right* where $U(x, y)$ is a formula in a linearly ordered structure $\mathcal{M}$ if

$$\mathcal{M} \vDash \forall x \forall y[(x < y \wedge U(x, y)) \to \forall z(x < z < y \to U(x, z))]$$

Suppose $\mathcal{N}$ is a linearly ordered structure. We say that $(A; C)$-*cut* in B in case of the splitting of $B \subset N$ into two convex subsets A and C ($A < C$; $A \cup C = B$). If A has a supremum or C has a infimum in $B \cup \{-\infty, \infty\}$, thereat the $(A; C)$-cut is said to be *rational*. Else, the $(A; C)$-cut is *irrational*. Sometimes, by $(A; C)$-cut we mean the next a number of formulas:

$$\{a \leq z \leq c | a\ from\ A, c\ from\ C\}.$$

Suppose $p\ from\ S_1(B)$. We say that a B-definable formula $\Psi(z, t)$ is *p-preserving* or *p-stable* if for every $\beta \vDash p$ there are $\alpha_1, \alpha_2 \vDash p$, that

$$\alpha_1 < \Psi(N', \beta) < \alpha_2.$$

In occation, where β is right (left) endpoint of set $\Psi(N', \beta)$ we say that formula is convex to the left (right).

If $\exists\ G(z, t, \bar{b})$ $(E(z, t, \bar{b}))$ formula which is the greatest p-preserving convex to the left (right) 2-B-formula thereat p is *semi-quasisolitary to the left (right)* where $p\ from\ S_1(B)$ is a non-algebraic type. It is called that $\bar{b}\ from\ A$. It means that for every p-preserving convex to the left formula $\Psi(z, t, \text{b})$, and for every $\beta \vDash p$ $\Psi(N', \beta, \bar{b}) \subseteq G(N', \beta, \bar{b})$

If type $p\ from\ S_1(B)$ is semi-quasisolitary to both sides then it is called to be *quasisolitary*.

Let $p\ from\ S_1(B)$ be quasisolitary. When the greatest convex to the left and to the right formula $F\left(z, t, \bar{b}\right) \equiv z = t$ then it is called that p is *solitary*

NOTATIONS AND ABBREVIATIONS

$\neg$	negation
$\forall$	universal quantifier
$\exists$	existential quantifier
$\rightarrow$	implication
$\wedge, \vee$	disjunction and conjunction
$\equiv$	elementary equivalence
$\mathcal{N}, \mathcal{M}, \ldots$	structures
$\prec$	elementary substructure
$\Leftrightarrow$	if and only if
$\vDash$	satisfaction in structure
$\Sigma, \mathbb{L}, \ldots$	languages
$C, D, \ldots$	sets
$c, d, \ldots$	elements of structures
$\bar{c}, \bar{d}, \ldots$	tuples
$N, M, \ldots$	universes of structures
$\beta, \gamma, \ldots$	elements of extensions of structures
$q(\bar{z}), p(\bar{t}), \ldots$	types
$\varphi(\bar{z}), \psi(\bar{t}), \ldots$	formulas
$C < D$	all elements of set C are less than any element of set D
$\|C\|$	cardinality of a set C
$\psi(N)(q(N))$	set of realizations of a formula ψ (type q) in $\mathcal{N}$
$V_p(\alpha), QV_p(\alpha)$	neighbourhood, quasi-neighbourhood of α in type p
$Th(\mathcal{N})$	$\mathcal{N}$ structure theory
$S_m(T)(S_m(C))$	set of any complete m-type (over set C) of theory T
$I(T, \gamma)$	number of models of cardinality γ of T
$tp(c\|C)$	type of c over the set C
$dcl(C)(acl(C))$	definable (algebraic) closure of a set C
$\perp^w$	relation of weak orthogonality of types
$\perp^a$	relation of almost orthogonality of types

INTRODUCTION

The research theme actuality. At present expansion of models by new relations is one of the primary directions of reseach in theory of models which is part of the mathematical logic.

The underlying theme in model theory is to classify first order theories. The first approximation in classifying was Shelah's notion of *stable theory*. Which recently has broadened and nowadays includes NIP theories.

Theory T is named to possess the independence property (IP), whenever it is possible to find a formula $\psi(\bar{z}, \bar{t})$ such that in every model $\mathfrak{M}$ of T for each $m < \omega$, a family of tuples $\bar{b}_1, \ldots, \bar{b}_m$ exists, to such an extent that it is easy to find a tuple $\bar{c}_J$ in $\mathfrak{M}$ such that $\mathfrak{M} \models \psi(\bar{c}_J, \bar{d}_j) \Leftrightarrow j \in J$ for every subset J of m. If it doesn't exist such formula, then T is said to have NIP, that is not the independence property.

Important part of investigating complete theories is to examine specifications of new relations necessary and/or sufficient to change class of model of complete theory in new signature or preserve it. One of the most significant classes of complete theories in NIP theories along with stable theories are o-minimal theories and a wider class including o-minimal theories - weakly o-minimal theories. This classes of theories are in the main scope of exploration of this work.

Leading specialists in model theory, such as B.I. Zilber, E. Hrushovski, A. Nesin, B. Poizat, G.Cherlin, J. Baldwin, E. Bouscaren, A. Wilkie, Ch. Steinhorn, D. Macpherson, D. Marker, B. Baizhanov, S. Shelah, M. Benedikt, A.Pillay, have received profound results in different problems of expansions.

J.T. Baldwin and K.Holland found sufficient conditions that there is model complete theory behind every unary ω_1-categorical expansion of strongly minimal model. D. Macpherson, Ch. Steinhorn and D. Marker have verified that an expansion of weakly o-minimal structure by particular type of convex unary predicate preserves weak o-minimality. B.S. Baizhanov has resolved problem of the weakly o-minimal theories expansion using unary convex predicate [8]. B. Sh. Kulpeshov presented the concept of convexity rank and obtained a description of weakly o-minimal theories in terms of definable sets of one-types convexity. Thesis concerns different classes of expansions of finite convexity rank weakly o-minimal theories which is quite new class of complete NIP theories.

The aim of the work is to examine issues of certain properties preservation (like quite o-minimality, weak o-minimality, countable categoricity, model completeness, convexity rank and others) in the process of expansion of models.

The objectives of the work are the following:

- Investigate inquiries of certain properties preservation of expantion of models by unary predicates.
- Investigate inquiries of certain properties preservation of expantion of models by equivalence relations.
- Investigate inquiries of certain properties preservation of expantion of models by arbitrary binary predicates.

The main states for the dissertation defense:

- Touchstone for maintaning aleph-nought categoricity in the process of weakly o-minimal expansion of a non-1-indiscernible weakly o-minimal aleph-nought categorical theory of convexity rank 1 by every single binary predicate.
- Touchstone for maintaning aleph-nought categoricity for a weakly o-minimal expansion of a 1-indiscernible weakly o-minimal aleph-nought categorical theory of convexity rank 1 by every single binary predicate.
- Maintaining weak o-minimality when expanding a weakly o-minimal ordered group by an externally definable binary predicate.
- Touchstone for keeping weak o-minimality and countable categoricity (and the 1-indiscernibility in addition to this). It is in next case. A weakly o-minimal 1-indiscernible countably categorical theory which has finite convexity rank is considered. A model of the theory is expanded using an relation of equivalence splitting the universe into infinite number of infinite convex classes.
- Maintaining quite o-minimality, countable categoricity and convexity rank when expanding a model of a quite o-minimal countably categorical theory by a convex unary predicates family which is finite.
- Maintaining the convexity rank and the countable categoricity under expantion of a theory model where theory is countably categorical weakly o-minimal with finite convexity rank and expantion is made by a convex unary predicates family which is finite.

The objects of research are complete NIP theories (theories lacking the independence property) and models of NIP theories. In particular, NIP theories include weakly o-minimal theories and stable theories.

The research subjects models of NIP theories, their properties and properties of these models under expansion by unary or binary predicates or equivalence relations

Methods of research: In the dissertation we use Classical Model Theory methods (in particular, method of quantifier elimination), inclusive of the ones which have been developed in model theory since 1980's and later. Among them we can note the methodology of investigating ordered structures, based on such notions as o-minimality and variants of o-minimality. In such cases it is typical to apply strict restrictions on sets definable by a formula which has the only free variable. Thus, if L is a language which includes language $L_0 = \{<\}$, where $<$ is a linear order on an o-minimal structure N and every single definable subset of the structure N is quantifier-free in L_0 then we can consider the o-minimal structure N as L-structure. It gives pattern for other determination. We set another unknown language instead language L_0. Then we regard L-structures to an extent so that the L_0-reduct is linear order or language of some stipulate type. And we want all definable subsets of structure N are L_0-definable (quantifier-free). It is possible to require for every model of the theory. Aside from that, we can note the methods of researching ordered structures developed in the last 20 years, such as describing models through analysis of behaviour of definable unary functions, the examine of models via systematization by convexity rank and others.

Scientific novelty of the dissertation research. Preservation of properties of expansion of models of complete theories such as NIP theory problem is unsettled at present. Contained in the survey classes of theories haven't been researched on an considered expansions.

Practical and theoretical research significance. Systematization of complete NIP theories stem from researches in this area. We can apply anticipated conclusions on the essence of expansions to the fields, rings and groups theories.

Dissertation thesis connection the with the another scientific investigation works. The following list shows scientific projects within framework of which PhD dissertation was carried out. Scientific projects of the program of grant financing of fundamental researches of the MES of Republic of Kazakhstan: "Properties of types in dependent theories" (2015-2017, 5125/GF4), "Basic and derived objects for ordered and generating structural objects as well as elementary theories" (2018-2020, AP05132546) and "Conservative extensions, countable ordered models and closure operators" (2018-2020, AP05134992).

Approbation of obtained results: PhD thesis results are tested at many foreign and domestic international scientific conferences and seminars:

- The 12th International Conference School "Problems Allied to Universal Algebra and Model Theory" (2017, Russia, Erlagol);
- The Sixth Congress of the Turkic World Mathematical Society (Astana, 2017);
- ASL European Summer Meeting "Logic Colloquium" (Udine, Italy, 2018);
- International Conference "Mal'tsev Meeting" (Novosibirsk, Russia, Institute of Mathematics, 2017, 2018);
- ASL North American Annual Meeting (Macomb, USA, Western Illinois University, 2018);
- The 6-thWorld Congress and School on Universal Logic (2018, France, Vichy);
- The Sixteenth Asian Logic Conference (Nur-Sultan, 2019);
- Annual International April Mathematical Conference (Almaty, Institute of Mathematics and Mathematical Modeling, 2017, 2018, 2019, 2020).

Publications: Research findings of the PhD thesis were published in 20 works, including 3 articles published in journals having a non-zero impact factor according to international databases Web of Science and (or) Scopus; 4 papers published in domestic journals recommended by CCFES of the Ministry of Education and Science of Kazakhstan. Also 13 abstarcts were published in materials of international scientific conferences.

Dissertation volume and structure. The dissertation consists of next units and structural item: page of title, contents, prescriprive references, abbreviations, notations and definitions, introduction, five sections (historical review, expansions of models by unary predicates, expansions of models by equivalence relations, expansions of models by binary arbitrary predicates, external definability and model completeness), inference and links. Dissertation's total number of pages equals 78. The work includes 94 references and four pictures.

Dissertation work main content. Section "Introduction" of the thesis includes the dissertation work aim, the research objectives of the thesis. It shows relevance of science research topic, scientific novelty of the research. There are the objects of investigation and the investigation subjects, primary states for the dissertation defence in the Introduction. The section explains practical and theoretical dissertation significance. There are described research methods. In the introduction there are given connection of the research thesis work with another scientific investigation works, approbation of obtained results, publications, as well as volume, structure and main content of the PhD thesis.

The first section describes the historical background and the present state of the model theory are under investigation.

The second section of the dissertation provides basic information and considers expansions of models by unary predicates.

The third section is devoted to expansions by equivalence relations of countably categorical, weakly ordered-minimal theories. Found a criterion for preserving countable categoricity and weak ordered-minimality.

In the fourth section of the dissertation considers arbitrary binary expansions of 1-indiscernible and non-1-indiscernible countably categorical models, weakly ordered-minimal theories of convexity rank 1.

The fifth section is focused on the class of externally definable expansions in the scope of preserving model completeness.

To fill out the section's main result we also show different examples of expansions which does not preserve certain properties.

The conclusion lists and generalizes the key conclusions reached in the PhD thesis.

1 HISTORICAL REVIEW

Throughout all the history the establishment of theory of models may be related to a number of directions. N model of signature Σ of first order predicate logic solves the complete elementary theory of given model, T equals to $Theory\ N$, it means the set of sentences of signature Σ, that holds in this model. We say *two models N_1 and N_2 of identical signature are elementary equivalent whensoever their elementary theories match or $Theory\ N_1$ equals to $Theory\ N_2$.* Investigations of elementary theory T were droven by four main pathways of research:

- Elementary theory decidability;

- Quantity of non-isomorphic models of complete theories;

- Elementary theory axiomatizability;

- Expansion of models by new relations.

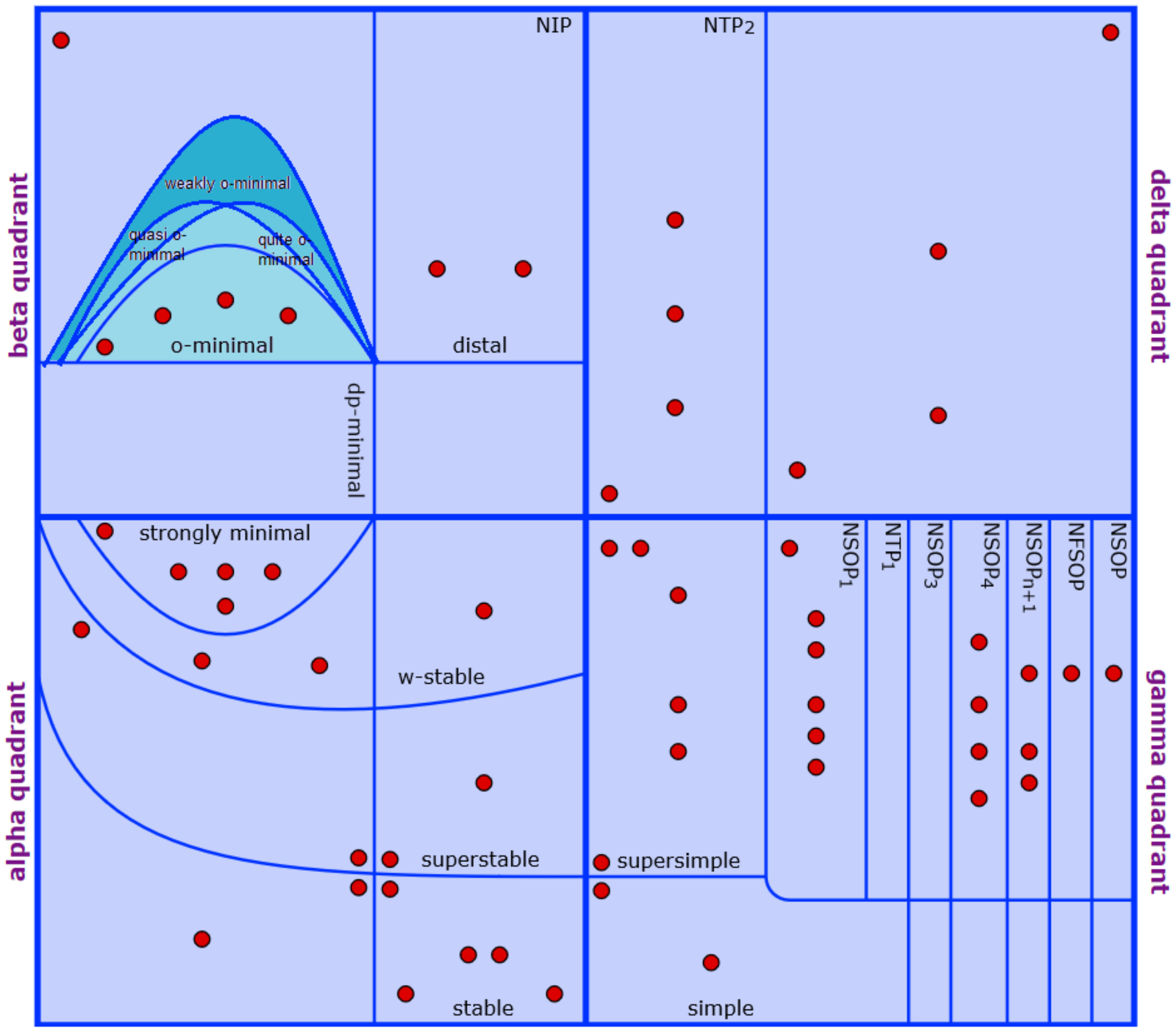
We state that a model $\langle M, \Sigma^+ \rangle$ is an expansion of a model $\langle M, \Sigma \rangle$ if $\Sigma \subset \Sigma^+$. The expansion is called an **essential expansion** if forbye there exists an n-ary Σ^+-formula $\varphi(x)$, to such an extent that the set $\varphi(M^n)$ is not definable over $\langle M, \Sigma \rangle$.

The dominant problem in this field of model theory is: for requisite features of the initial model, to obtain a conditions for the new relations, with the purpose that under expansions by these relations requisite features are preserved.

These features can be hereinafter: model completeness (for example expansion of weakly o-minimal model complete theories), expanded model elementary theory decidability (for instance, exponential function expansion of the theory of the real numbers field, open question), strong minimality, omega-stability, superstability, stability, o-minimality, weak o-minimality, finite cover property, no independence property, and others.

Researches in second half of the preceding century occasioned in the complete theories segregation into classes, subject to the nature of definable sets and their systems.

In the preceding decades expansion challenges are suitable for all classes of complete theories. Systematization of complete theories is displayed in Picture 1:



Picture 1[1] - Complete Theories

Major experts in model theory got extensive results in distinct challenges of expansions (expansion by elementary substructures, automorphisms, nonelementary substructures, non-indiscernible sets etc.).

Strongly minimal theories (B.I. Zilber, E. Hrushovski, A. Baduisch, B. Poizat, E.A. Palutin, J. T. Baldwin, K. Holland, V.V. Verbovskiy, S. Buechler, A.T. Nurtazin, A. Pillay, M. Macintyre, B. Baizhanov - J. Baldwin etc.).

ω-stable (A. Nesin, A. Borowik, B. Poizat, G. Cherlin, B. Zilber, J. Baldwin - K. Holland, A. Baduisch and others).

Superstable (E. Bouscaren, T.G. Mustafin, B. Poizat, B Baizhanov - B. Baldwin - S Shelah, E. A. Palyutin, A.A. Stepanova and others).

Stable (B. Poizat, E. Bouscaren, J. Baldwin - M. Benedikt, Kazanova - Ziegler, B. Baizhanov - J. Baldwin, K. Kudaibergenov and others).

O-minimal (L. Van den Dries, A. Wilkie, Ch. Steinhorn, A. Pillay, D. Marker, D. Macpherson, E.A. Palyutin, S. Starchenko, Peterzil, B. Baizhanov, E. Baisalov -

[1] http://www.forkinganddividing.com/

B. Poizat and others).

Weakly o-minimal (D. Macpherson – D. Marker – Ch. Steinhorn, E.A. Palyutin, B.S. Baizhanov, V.V. Verbovskiy, B.Sh. Kulpeshov, R.D. Arephyev, R. Vencel and others).

Quasi o-minimal (O. Belegradek - A. Strobushkin - M. Taitslin, J. Baldwin - M. Benedikt and others).

Quite o-minimal (B. Kulpeshov, V.Versbovskiy).

Dependent (NIP) theories (S. Shelah, H.D. Macpherson – D. Marker – Ch. Steinhorn, J. Baldwin, M. Benedikt, A. Pillay, F. Wagner, V. Verbovskiy and others).

Simple (S. Shelah, E. Hrushovski, E.A. Palyutin, B. Poizat, N. Kim, A. Pillay, M. Macintyre, F. Wagner, V. Kolesnikov, Vasilyev - Itay, Pillay - Poltakovskaya and others).

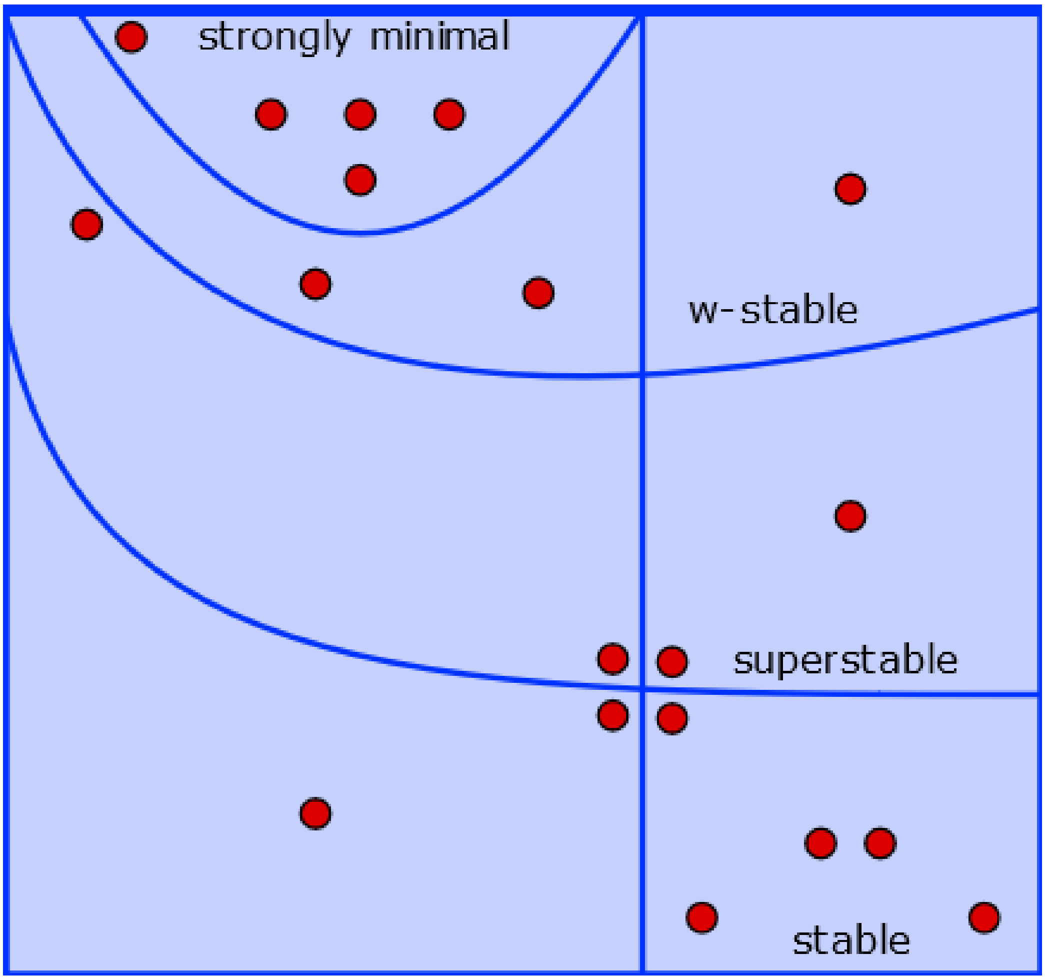
E^*-stable (E.A. Palyutin, T.G. Mustafin, B. Poizat, T. Nurmagambetov, A. Stepanova and others).

Numerous outcomes were revealed in the upcoming classes of complete theories: dependent theories, weakly o-minimal theories, o-minimal theories, stable theories, superstable theories, o-stable theories, strongly minimal theories.

The upcoming approach the expansion problem can determined:

Let C_1, C_2 be individual classes of complete theories, if $\langle M, \Sigma \rangle$ is a model of some complete theory C_1, then which considerations on new relations are necessitous and/or sufficient for the purpose of having $\langle M, \Sigma^+ \rangle$ be a model of theory C_2?

Consider results in alpha quadrant in Picture 2.



Picture 2 – Alpha Quadrant

Strongly minimal theories.

A theory is titled *strongly minimal* if in every single model every definable set is finite or negation of this definable set is finite (1971) [9]. A problem of unary function expansions of an algebraically closed field, to such an extent that this function is an automorphism was examined by A. Macintyre. B. Zilber began to investigate the systematization of strongly minimal theories as chunk of a investigation of the uncertainty of finite axiomatizability and the spectrum of complete theories. Zilber suggested a hypothesis on the geometry of strongly minimal theories. Hypothesis: For strongly minimal theories the geometry arising from the algebraic closure operation is either one of the upcoming types: expansions of an algebraically closed field, trivial or locally modular. E. Hrushovski (1988) built an illustration of a strongly minimal non-locally-modular theory, so that this theory can not be interpreted in a field, and in the theory a group is not interpreted. That is a counterexample to Zilber's hypothesis. V.V. Verbovskiy verified for that example that its elementary theory doesn't accept exclusion of imaginaries [10, 11] (2002, 2006). In 2004 B.S. Baizhanov and J.T. Baldwin showed: for all strongly minimal theory the upcoming holds true: expansion by an arbitrary set is stable (superstable) whenever the strongly minimal formula has trivial geometry [12]. However in 2004 J.T. Baldwin and K. Holland established possibility of unary predicate expansion of an algebraically closed field to such an extent that structure obtained after expansion would be omega-stable of Morley rank κ for any natural κ [13].

Omega-stable theories.

Bruno Poizat in his research articals published in The Journal of Symbolic Logic in 1999 and 2001[14, 15] constructed finite Morley rank omega-stable field, which has two arbitrary unary predicates. John T. Baldwin and Kitty Holland found out that it has non-model-complete teory. In their 2004 work published in journal "Annals of Pure and Applied Logic" [13, P. 159] J.T. Baldwin and K. Holland revealed sufficient requestes, that a strongly minimal model has a model complete theory in case by unary predicates for every ω_1-categorical expansion. In example constraction Hrushovski put notions "pre-dimension and dimension" of finite structures as foundation. When he built omega-stable theory, of a finite extension of the designated finite structure, Hrushovski determined dimension as pre-dimension. In 2003 premised on the conception of topological space completion, V.V. Verbovskiy has developed a pre-dimension defining technique on one class of infinite structures, which has enabled to transfer to the study of generic stable structures on the manner of studying generic omega-stable structures [16].

Superstable theories.

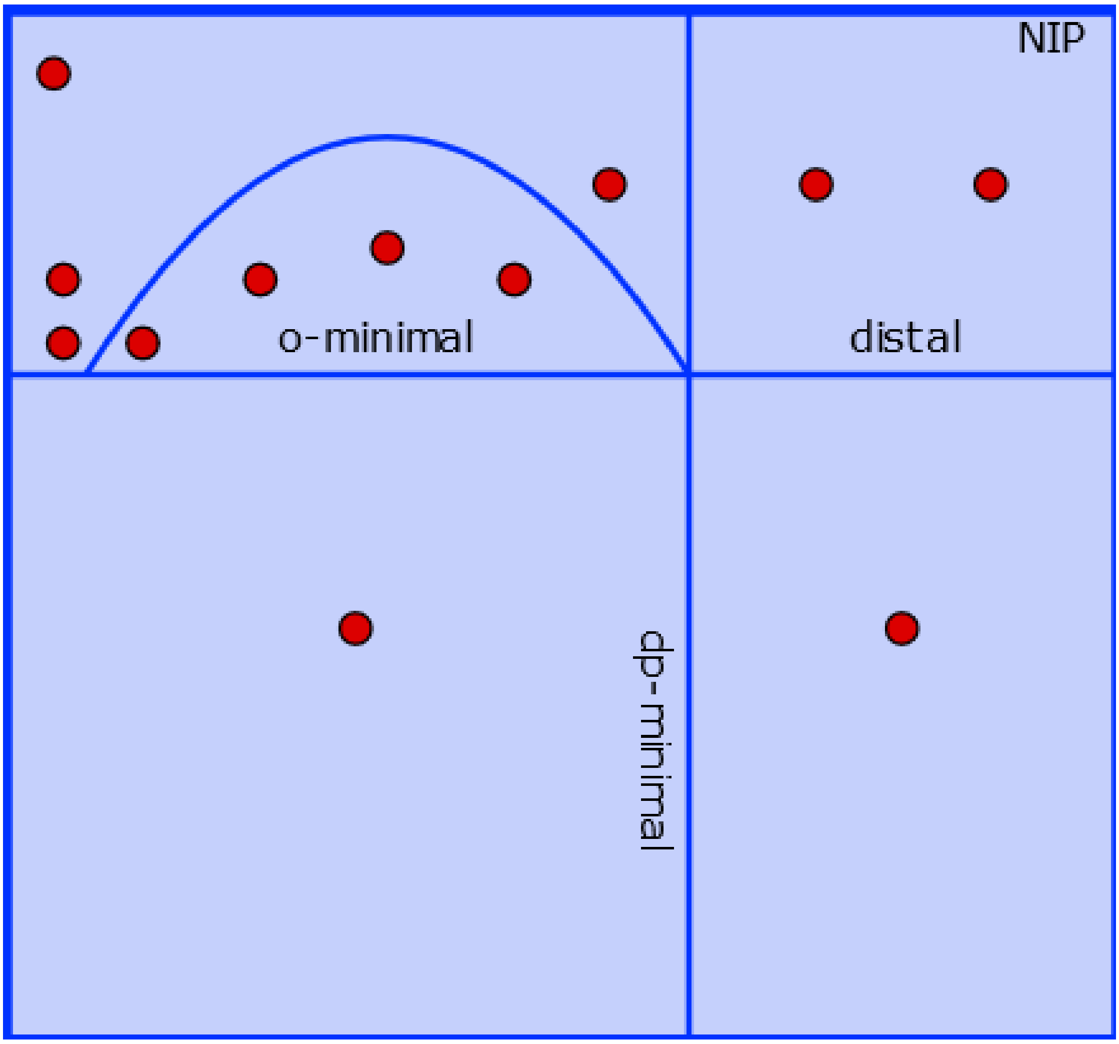
For superstable theories Elisabeth Bouscaren examined sufficient expansions of models by unary predicate establishing an elementary substructure.

In 1989 she verified that an absense of the dimension order property in the initial superstable theory implies stability (superstability) of a pair of models is stable (superstable) and vice versa [17]. In 1988 E. Bouscaren and B. Poizat verified that in stable theories this outcome does not possess [18], by building an example of non-superstable theory, where pairs of models theory is complete and stable and dimension order property possess. In 1989 part of Bouscaren's [17, P. 205] evidence was that she verified that in the expanded language the types identity over the two tuples follows from the types of the tuples in small model identity in the original language. Suchlike characteristic is considered being *benign*. In 2005 J. Baldwin, B.S. Baizhanov and S. Shelah have verified that sameness of strong types of the original language tuples over random set of superstable model causes sameness of types over expanded language tuples [19]. Such characteristic is considered being *weakly benign*. It is thus confirmed weakly benignity of each and every superstable theory sets.

Stable theories.

Couples of stable theory model, to an extent that large model is satiated over a submodel were examined by Bruno Poizat. He named such a couple of models lovely pair. In 1983 he verified that there is no finite cover property in the initial theory whenever the theory of lovely pairs is complete [20]. In 2000 steadiness of a satiated expansion of model of stable theory by predicate, which discern a non-definable set were proved by Baldwin and M. Benedikt [21]. Results from [21, P. 4937] were enhanced by E. Casanovas and M. Ziegler in 2001, by getting rid of the indiscernibility, and establishing non finite cover property in terms of this set [22]. The outcomes of Baldwin-Benedikt-Casanovas-Ziegler and the outcomes B. Poizat and E. Bouscaren of were enhanced by B.S. Baizhanov and J. Baldwin, by verifying that the weakly-benign set constraint of an expanded model has stable theory whenever the weakly benign expansion of model preserves stability (resemblant outcome takes place for the class of superstable and ω-stable theories as well). Moreover in 2004 Casanovas-Ziegler's question on designation of the characteristics of the model $\mathfrak{M}$ having finite cover property in regards to set A was answered by B.S. Baizhanov and J.T. Baldwin [12, P. 1243].

Consider the beta quadrant in Picture 3.



Picture 3 – Beta Quadrant

O-minimal theories.

The dynamic investigation of linearly ordered theories founded on the concept of o-minimality has started since the middle of 80's. In 1996 one of the significant results is real numbers field unary expansion, which is o-minimal, decidable and admits quantifier elimination, by exponential function has a model completeness property and it's theory is o-minimal was verified by Alex Wilkie [23]. In 1997 van den Dries, A. Macintyre, D.Marker showd that every single o-minimal structure on $\mathbb{R}$ produces a family of definable sets that has local triviality, stratification and property of uniform finiteness and is quite stable under various topological / geometrical operations [24]. The property that every single structure elementary equivalent to a structure linearly ordered is o-minimal whenever a structure linearly ordered is o-minimal as well verified by Ch. Steinhorn, A. Pillay and J. Knight. Furthermore in 1986 and 1988 they clarified definable functions characteristics [25-27]. In 2007 K.Zh. Kudaibergenov enhanced the Marker's outcomes in o-minimal theories about small extensions of models in [28]. In 2007 B.S. Baizhanov verified that there exists an elementary extension to extent that the unary partial functions class with definable parameters is not identical to the unary partial functions class, defined over original language parameters whenever ordered-minimal expansion of a T theory model that is dense ordered-minimal and accepts QE is essential [29].

Let $\mathbb{M} = \langle M, \Sigma \rangle$, $A \subset M$ to such an extent that $A \neq \Theta(M, \bar{a})$ for every single definable set of structure $\mathbb{M}^{+} := \langle M, \Sigma \cup \{P\} \rangle$. When investigating new formulas of the expanded model for the formula $\varphi(x, \bar{y})$ of the initial language, the formula of the expanded language $\exists x(\neg P(x) \wedge \forall \bar{y}(\wedge_i P(y_i) \rightarrow \varphi(x, \bar{y})))$ broaches two questions linked to the φ-type of the model of the original theory:

Will the set $\{\varphi(x, \bar{a}) | a \in A\}$ be consistent?

Is the φ-type over the set A be realized in the model?

In some cases, positive answers to these questions suggest the nature of formulas of the expanded model: lovely pairs of models (B. Poisat [20, P. 239]), small indiscernible set (Baldwin-Benedikt [21, P. 4937]), small set without the finite cover property with respect to this set (Casanovas-Ziegler, [22, P. 1127]), lovely pairs (Pillay-Vassiliev [30]), H-structures (Berenstein-Vassiliev [31]).

The notion of geometric theories were originated by E. Hrushovski and A. Pillay in [32] (1994). It is a regular comprehensiveness of the classes of dense o-minimal and strongly minimal theories in as much as it admits the elimination of "there are infinitely many" quantifier property and the algebraic closure exchange property, which is geometric theory. Since 2010 A. Berenstein and E. Vassiliev have been studying unary predicate expansions of geometric theories, such that predicate has extension and density properties as well as investigating interconnections of the properties of the initial theory and the properties of expanded theories [31, P. 866; 33-34]. Their work is basen on the concept of simple theory models – lovely pair. Belle pair – lovely pair of models of a simple theory was examined by Bruno Poizad. In [30, P. 491; 35-38] A. Berenstein and E. Vassiliev developed and studied it.

Weakly o-minimal theories.

The French-Kazakh Colloquium on model theory in Almaty in June 1994 was accompanied by extensive amounts of prominent scientists. An american scientist Charles Steinhorn pontificated on the Colloquium with a report on o-minimality. Since then the collaboration of B.S. Baizhanov and other Kazakh colleagues with Ch. Steinhorn arose.

Charles Steinhorn has sent article draft of D. Macpherson, D. Marker, Ch. Steinhorn [39] on weak o-minimality and a parcel of copies of works on o-minimality by J. Knight, A. Pillay, C. Steinhorn [25, P. 565; 26, P. 593; 27, P. 469], L. Mayer, D. Marker, Ch. Steinhorn, A. Pillay, [40-43] Kazakh scientists resolved all the problems stated in the works on weak o-minimality [39, P. 5435].

External definability.

D. Macpherson, D. Marker, Ch. Steinhorn approach. In 1994 it was verified by D. Macpherson, D. Marker, Ch. Steinhorn (in the draft [39, P. 5435]) that for an o-minimal structure expansion by predicate preserves weak o-minimality in case that such predicate is “unary convex predicate” and it is crossed by a uniquely realizable cut. According to D. Marker [40, P. 63] 1-type uniquely realizable over a model M, that is if for every single α realizing this 1-type p, α is the only realization of p in $Pr(M \cup \{\alpha\})$, has the corresponding characteristic: there is no definable function operating on the type’s set of realizations. Two structures simultaneously was considered by Macpherson-Marker-Steinhorn: $\mathfrak{K}^+ = \langle M; \Sigma \cup \{U^1\} \rangle$ and $\mathfrak{L} = \langle N; \Sigma \rangle$, that is a satiated $\mathfrak{M}$ elementary extension and a model of o-minimal theory. A new unary convex predicate U was defined with an element $\alpha \in N \backslash M$ that is the realization of type $p \in S_1(M)$, an irrational $1-$type, to such an extent that the following holds for every *d from M:*

$$\mathfrak{K}^+ \vDash U(d) \Leftarrow \mathfrak{L} \vDash d < \alpha.$$

The signature $\Sigma^+ = \Sigma \cup \{U^1\}$ formulas $\gamma(\bar{y})$ are constructed by induction. It implies a signature Σ formula $K_\gamma(\bar{y}, \alpha)$ exists to an extent that for any $\bar{d}$ *from* M the following holds:

$$\mathfrak{K}^+ \vDash \gamma(\bar{d}) \Leftarrow \mathfrak{L} \vDash K_\gamma(\bar{d}, \alpha).$$

The occasion $\gamma(\bar{y}) = \exists x \psi(x, \bar{y})$ was the decisive point in this construction. They defined

$$K_{\exists x \psi(x,\bar{y})}(\bar{y}, \alpha) := \exists z_1 \exists z_2 \exists x (z_1 < \alpha < z_2 \wedge \forall z (z_1 < z < z_2 \rightarrow K_{\psi(x,\bar{y})}(x, \bar{y}, z)).$$

Although 1-type p from $S_1(M)$ is uniquely realizable, 1-formulas over $M\alpha$ convex to the right and the left from α have solutions outside $p(\mathfrak{L})$. Hence if $\mathfrak{L} \vDash K_{\exists x \psi(x,\bar{y})}(\bar{d}, \alpha)$, so for certain $b_1, b_2 \in M$ for every single $\bar{d}$ *from* M,

$$\mathfrak{L} \vDash \exists x (b_1 < \alpha < b_2 \wedge \forall z (b_1 < z < b_2 \rightarrow K_{\psi(x,\bar{y})}(x, \bar{d}, z)).$$

It means the previous formula segment holds on $\mathfrak{K}$, namely

$$\mathfrak{K} \vDash \exists x \forall z (b_1 < z < b_2 \rightarrow K_{\psi(x,\bar{y})}(x, \bar{d}, z))$$

Hence there exists c *from* M to such an extent that

$$\mathfrak{K} \vDash \forall z (b_1 < z < b_2 \rightarrow K_{\psi(x,\bar{y})}(c, \bar{d}, z)).$$

Consequently $K_{\psi(x,\bar{y})}(c, \bar{d}, z) \in p$.

So for every single Σ^+-M-1-formula $\phi(x, \bar{a})$ the set of all its realizations in $\mathfrak{K}^+$

will be a convex sets finite union, $\gamma(\mathfrak{K}^+, \bar{d}) = K_\gamma(\mathfrak{L}, \bar{d}) \cap M$, owing to the fact that $K_\gamma(\mathfrak{L}, \bar{d})$ consists of finite union of points and intervals. Elementary theory of $\mathfrak{K}^+$'s is weakly o-minimal being that the number of convex sets is not infinite and because of this doesn't depend on parameters.

There was made a systematization of the non-orthogonality of 1-types theory in [25, P. 565; 40, P. 63; 41, P. 146; 42, P. 185]. On this basis in 1995 it was found out that the case of $p\ from\ S(M)$ is type that is non-uniquely realizable. (Pillay-Steinhorn, D. Marker, L. Mayer, Marker-Steinhorn, 1986–1994), B.S. Baizhanov proffered [44] to take the constants for $K_{\exists x\psi(x,\bar{y})}$ from an indiscernible infinite sequence $\langle\alpha_n\rangle_{n<\omega}$ over M where α_n is from $p(\mathfrak{L})$. When $K_{\psi(x,\bar{y})}(\mathfrak{L}, \bar{d}, \bar{\alpha}_n) \cap M = \emptyset$ there exists a finite number of irrational cuts in other words one-types over M, to such an extent that for every single such one-type $r \in S_1(M)$, the subset $K_{\psi(x,\bar{y})}(\mathfrak{L}, \bar{d}, \bar{\alpha}_n)$ of

$QV_r(\bar{\alpha}_n) := \{\beta \in r(\mathfrak{L})|$ there exists an $A \cup \{\bar{\alpha}_n\}$-1-formula $\Theta(x, \bar{\alpha}_n)$, such that

$$\beta \in \Theta(\mathfrak{L}, \bar{\alpha}_n) \subset r(\mathfrak{L})\}.$$

There was obtained the one-types systematization over "weakly o-minimal theory" model subset by B.S. Baizhanov in 2001 [8, P. 1382]. He solved a problem of expanding a "weakly o-minimal theory" model by an unary convex predicate. The monotonicity characteristic for definable functions on weakly ordered-minimal structures was verified by R.D. Aref'ev in 1997 [45]. Sample of a weakly ordered-minimal structure, whose theory is not weakly ordered-minimal was built by Macpherson-Marker-Steinhorn [39, P. 5435]. Also, in 2001, "a weakly o-minimal" ordered group example, which theory isn't "weakly o-minimal" was built by V.V. Verbovskiy [46]. In 1998 B.Sh. Kulpeshov developed a designation of a linearly ordered structure weak ordered-minimality in terms of the set of realizations 1-types convexity in the examining of "weakly ordered-minimal structures" [47], and he made a complete characterization of linear orders that is weakly o-minimal, he established the concept of unary formula convexity rank , that is useful in investigating countably categorical structures. Also in 2007 and 2011 he found a touchstone for binarity of weakly o-minimal countably categorical structures in expressions of types binarity and convexity rank [48, 49]; In 2006 he proved the binarity and characterized the structures of convexity rank 1 that are countably categorical weakly o-minimal [50, 51]; Also in 2011 and 2013 he characterized quite o-minimal countably categorical structures [52, 53]. There was introduced a designation of behavior of p-preserving to the left convex and to the right convex 2-formulas in "weakly o-minimal theories" by Bektur S. Baizhanov and Beibit Sh. Kulpeshov in 2006 [54]. Starting from 2006 B.Sh. Kulpeshov in the series of works has given the criterion for binarity of "weakly o-minimal" countably categorical theories in the point of view of "convexity rank" and binarity of every non-algebraic 1-type, a complete characterization of countably categorical "weakly ordered-minimal finite rank of convexity" theories, and an entire countably categorical "quite o-minimal theories" characterization [49, P. 354; 51, P. 185; 52, P. 387; 55-56] (2006-2016).

Since 2018 successive results in a number of work on the field of expansions of "weakly o-minimal structures" were published by B.Sh. Kulpeshov and S.S. Baizhanov [7, P. 673 ;57-58]. In [57, P. 106] B.Sh. Kulpeshov and S.S. Baizhanov proved that convexity rank and $\aleph_0$-categoricity of an expansion of a "weakly o-minimal" $\aleph_0$-categorical theory of finite "convexity rank" is preservated in case of finite set of unary convex predicates expansion. In [58, P. 207] authors have developed a criterion for preserving both weak o-minimality and $\aleph_0$-categoricity in case by an relation of equivalence expanding of 1-indiscernible weakly ordered-minimal $\aleph_0$-categorical structures. In [58, P. 207] they also developed a criterion for preservation of the $\aleph_0$-categoricity of a weakly o-minimal 1-indiscernible expansion by a binary predicate on countably categorical weakly o-minimal 1-indiscernible structures of rank-1 convexity. In [7, P. 673] it was developed a touchstone the countable categoricity of a weakly ordered-minimal expansion of 1 rank of convexity under expansion by every single binary predicate of weakly o-minimal non 1-indiscernible countably categorical structures is preserved.

The concepts of a weakly quasi-o-minimal model and theory were established and inspected by K.Zh. Kudaibergenov [59] (2010). In 2012 and 2013 K.Zh. Kudaibergenov established and inspected numerous o-minimality extensions into "partial orders" [60, 61]. It was proceeded generalizations of the o-minimality concept by various ways by K.Zh. Kudaibergenov. In 2018 K.Zh. Kudaibergenov established and inspected the notions of right o-minimality, multi-R-minimality, and their variants [62].

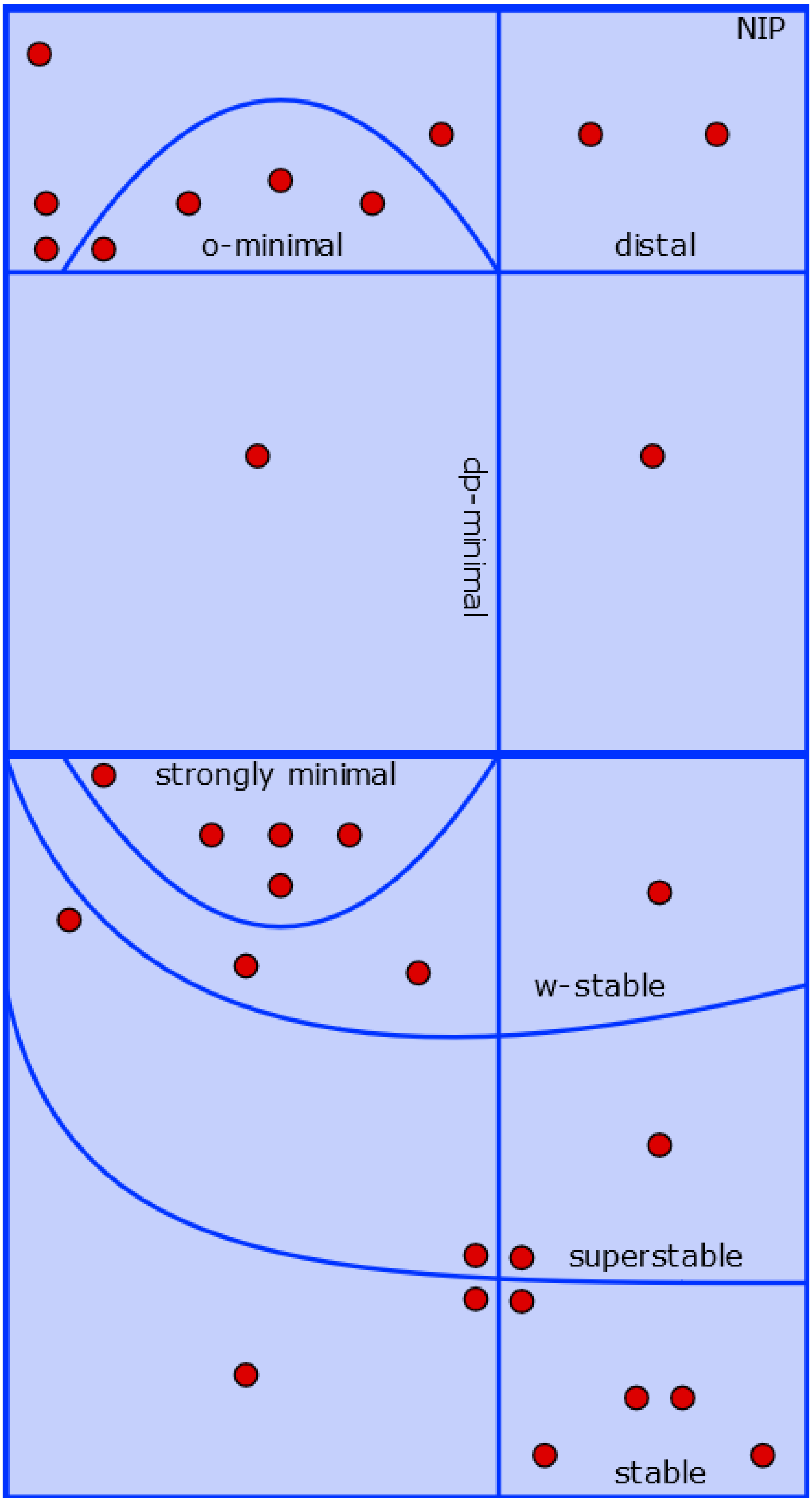
"Countably categorical weakly ordered-minimal theories" were investigated by H.D. Macpherson, B. Herwig, A.T. Nurtazin, G. Martin and J.K. Truss and they verified that every 3-indiscernible model is n-indiscernible for every single natural n. They built samples to such an extent that there is a 2-indiscernible model, which is not 3-indiscernible [63] (1999).

In a series of papers on the field of circularly ordered structures B.Sh. Kulpeshov (2006-2016) [64-67] obtained a number of results. B.Sh. Kulpeshov fetched complete designation of the behavior of functions definable unaryly for a countably categorical 1-transitive structure weakly circularly minimal [64, P. 555]. He characterized up to binarity countably categorical "weakly circularly minimal structures" with a non-primitive 1-transitive automorphism group [65, P. 282]. B.Sh. Kulpeshov developed a touchstone that 1-type of convexity rank 1 realizations is indiscernible in non-1-transitive "weakly circularly minimal" countably categorical structures [66, P. 255]; He defined almost binarity of countably categorical non-1-transitive "weakly circularly minimal theories" [67, P. 38].

There was established characterization up to binarity of "weakly circularly minimal structures" countably categorical with a primitive automorphism group by B.Sh. Kulpeshov and H.D. Macpherson [68] in 2005. High homogeneity of every 6-homogeneous weakly circularly minimal countably categorical structure was verified by them. In 2015 an any weakly circularly minimal cyclically ordered group is abelian was verified by V.V. Verbovskiy and B.Sh. Kulpeshov [69]. There was acquired a complete description of “weakly circularly minimal” non-1-transitive “countably categorical n-convex” (where $n > 1$) almost binary of convexity rank 1 theories by A.B. Altayeva and B.Sh. Kulpeshov [70] in 2016.

Type $p \in S_n(A)$ *is named definable if for every formula* $H(\bar{x}, \bar{y})$ *there is a controlling formula* $d_H(\bar{y}, \bar{\alpha})$ *that* $H(\bar{x}, \bar{a}) from\ p \Leftrightarrow d_H(\bar{a}, \bar{\alpha})$ *holds*. Definability of every single type over every set is the key characteristic of a stable theory was acquired in 1978 by Shelah [71], that is, for any type $p \in S(B)$ and for every formula $H(x, \bar{y})$ there is a controlling formula $d_H(\bar{y}, \bar{a})$ if and only if an elementary theory is stable. Extension is said to be *n-conservative* if the type of any n-tuple of elements of L over K is definable where $\mathfrak{L}$ is an elementary extension of a model $\mathfrak{K}$. There was verified by L. van den Dries [72] in 1984 that any type over the field of real numbers is definable. And it is true that every elementary extension is conservative. In 1994 this result was applied and enhanced to the class of “o-minimal theories” by David Marker and Charles I.Steinhorn. They verified that from 1-conservativity of a couple of models of an “o-minimal theory” it folows verifying their n-conservativity [42, P. 185]. At a late date Anand Pillay re-proved this result. He realized that the “o-minimal theory” is axiomatizable for conservative pairs of models [43, P. 1400]. This result relates to the class of “weakly o-minimal theories”, however in 2005 it was disseminated to a broader class by B.S. Baizhanov. He constructed non-2-conservative pair of models of “a weakly minimal theory” that is 1-conservative [73, 74]. In 2007 B.S. Baizhanov established the conservative pairs models of “weakly o-minimal theories” axiomatizability condition and verified that for any model of “a weakly o-minimal theory”, excluding an discrete linear order theory o-minimal expansion by maximal and minimal elements, there exists a saturated conservative elementary extension [75].

Consider a wider class of complete theories, which is union of alpha and beta quadrant in Picture 4.



Picture 4 – NIP Theories

Dependent theories (NIP theories).

A problem of systematization of dependent theories appeared since the end of 90s. Proper subclasses (“weakly o-minimal”, “o-minimal”, and “quasi-o-minimal” classes of complete theories) of the dependent theories class were obtained [42, P. 185; 43, P. 1400]. B.S. Baizhanov and V.V. Verbovskiy, Based on the methodology developed in stable theory, ascertained the class of ordered stable theory. In 2011 verified that the o-stable theory class is a dependent theory's subclass and that the pure linear order theory is o-superstable [76]. Definability of one-types for o-stable theories was explored in [77] in 2015 by B. Baizhanov and V. Verbovskiy. O-stable ordered groups and fields were investigated by V.V. Verbovskiy. He verified o-stable ordered group commutativity. Additionally, he characterized definable subsets, and constructed a vast amounts of non-trivial o-stable ordered groups examples [78] (2012). S. Shelah represented the notion of a dp-minimal theory in the frame of systematization of dependent theories . There was verified of o-stablity for "dp-minimal theories" with a definable linear order by V.V. Verbovskiy [79] in 2010. There was proposed the notion of "a theory stable up to Δ" by Viktor V.Verbovskiy [80] in 2013. He verified that NIP theories are "stable up to a certain formulas subset" without the independence property [80, P. 119].

In 2013 using stability up to Δ it a characterization of NIP theories was shown and the concept of relative stability was introduced by V.V. Verbovskiy [80, P. 119]. He proceeded to investigate definability of types and relatively stable theories in [81] and verified that for a stable up to delta theory T it holds that its delta part is definable if and only if every single one-type over a model of T is definable. V. Verbovskiy proceeded to study ordered o-stable groups in 2018. He built a sample of an ordered group with "Morley o-degree" at most 4 of Morley o-rank 1 and showed that every Morley o-rank 1 ordered group with definable convex subgroups, that is boundedly many, is "weakly o-minimal" [82]. In 2015 and 2018 circularly ordered groups were investigated in the articles [69, P. 82] and [83] respectively. The Abelian property of weakly circularly minimal groups was first checked by Beibit Kulpeshov and Viktor Verbovskii. Next it was amplified up to the circularly ordered class of o-stable groups by V. Verbovskiy.

The first-order theories independence property was investigated by K.Zh. Kudaibergenov. In 2011 he disproved the existence of infinite indiscernible sequences of big cardinalities models of NIP theories, that is the Shelah's hypothesis strong form [84]. In 2013 he refuted the Adler's claim. Kanat Zh. Kudaibergenov built a NIP theory, in which atomic formulas without independence property [85].

2 EXPANSIONS OF MODELS BY UNARY PREDICATES

Definition 2.1 If every M definable subset is a finite "union of convex sets" then totally ordered structure $(M,<,\ldots)$ is called *"weakly o-minimal"* [8, P. 1382].

Definition 2.2 [8, P. 1382] If M is "a weakly o-minimal structure", $A, B\ are\ subsets\ of\ M,\ M$ is $|A|^{+}$-satiated, $p, q\ from\ S_1(A)$ are non-algebraic 1-types then it is said that type p is "*non weakly orthogonal" to* q ($p \not\perp^{w} q$) if there exists a formula $H(x,y,\bar{b}), \bar{b}\ from\ A$ and a realisation $\beta\ from\ p(M)$ that there exists realisations $\alpha_1, \alpha_2\ from\ q(M)$ and the following holds: $\alpha_1\ from\ H(M,\beta)$ and $\alpha_2\ from\ \neg H(M,\beta)$.

Lemma 2.1 [8, P. 1391] *The relation of "non-weakly orthogonality" is an equivalence relation on* $S_1(B)$ *for B subset of K, where K is a "weakly o-minimal structure"* .

Proof: Reflexivity is obvious, for any type $p \in S_1(B)$ it is non-weakly orthogonal to itself: if we consider $\alpha \in p(K)$ we can use formula $H_0(x,z) := x = z$, in that case there are $\alpha_1, \alpha_2\ from\ p(K)$ such that $K \vDash H_0(\alpha_1, \alpha)$ and $K \vDash \neg H_0(\alpha_2, \alpha)$, where α_2 is any other realisation of $p(K)$ and $\alpha_1 = \alpha$. Let $p, q \in S_1(B)$ be two 1-types such that $p \not\perp^{w} q$. Then $p(x) \cup q(z)$ is not complete type. Thus $p(x) \cup \beta$ such that $\beta \in q(K)$ is not complete type. Then $q \not\perp^{w} p$. Let $p, q, r \in S_1(B)$ be 1-types such that $p \not\perp^{w} q$ and $q \not\perp^{w} r$ then $p \not\perp^{w} r$ as if we consider formula $H_1(x,z)$ and $H_2(x,z)$ such that $\beta_1 \in H_1(K,\alpha)$ and $\beta_2 \in \neg H_1(K,\alpha)$ and $\gamma_1 \in H_2(K,\beta)$ and $\gamma_2 \in \neg H_2(K,\beta)$ where without loss of generality β can be either β_1 or β_2. Let β be equal to β_1. Then if we conisder a formula $H_3(x,z) := \exists y (H_2(x,y) \wedge H_1(y,z))$ we get $K \vDash H_3(\gamma_1, \alpha)$ and $K \vDash \neg H_3(\gamma_2, \alpha)$ which is $p \not\perp^{w} r$

■

In [39, P. 5435] some concepts originally were introduced. We recall them. Suppose following statements hold: $Y\ subset\ of\ K^{n+1}$ is an $\emptyset$–definable set; $\pi: K^{n+1} \to K^{n}$ is a projection in which the last coordinate throw out; $Z := \pi(Y)$; for every $\bar{a}\ from\ Z$ takes place

$$Y_{\bar{a}} := \{z : (\bar{a}, z)\ from\ Y\}.$$

Let the set $Y_{\bar{a}}$ bounded above, but has no supremum in K for every $\bar{a}\ from\ Z$. We denote "$\sim$" as a relation of $\emptyset$–definable equivalence on K^{n}, and define it by the next way:

$$\bar{c} \sim \bar{d}\ \ for\ \ each\ \ \bar{c}, \bar{d}\ from\ K^{n} \setminus Z,\ \ and\ \ \bar{c} \sim \bar{d}\ \ if\ only\ if\ \sup\ Y_{\bar{c}} = \sup\ Y_{\bar{d}},$$

$if\ \ \bar{c}, \bar{d}\ from\ Z$. Suppose that $\overline{Z} := Z/\sim$, and as $[\bar{c}]$ we denote the $\sim$-class of the tuple $\bar{c}$ for every tuple $\bar{c}\ from\ Z$. It exists a natural order $\emptyset$–definable linear on $K \cup \overline{Z}$, which determined next way. Let $\bar{c}\ from\ Z$ and $a\ from\ K$. Then $[\bar{c}] < a$ $if\ only\ if\ \ w < a$ for any $w\ from\ Y_{\bar{c}}$. If $\bar{c} \nsim \bar{d}$, then there is some $x\ from\ K$ that

$[\bar{c}] < x < [\bar{d}]$ or $[\bar{d}] < x < [\bar{c}]$ and thus " $<$ " induces a linear order on $K \cup \overline{Z}$. The set $\overline{Z}$ we call a *sort* in $\overline{K}$ (a $\emptyset$–definable sort in $\overline{K}$ in this exact case), where $\overline{K}$ is a Dedekind structure K completion. And $\overline{Z}$ is considered as naturally embedded in $\overline{K}$. The same way, it possible to get a sort in $\overline{K}$, taking into consederation infinum instead of supremum.

Definition 2.3 [39, P. 5435] Function $f: B \to L$ is called locally decreasing (locally constant or locally increasing) on B where B is infinite subset of N, L is subset of $\overline{N}$ and N is structure linearly ordered if for each single $b \in B$ there exists an infinite interval $J\, from\, B$ containing b on which f is strictly decreasing (constant or strictly increasing).

If f is locally decreasing or locally increasing on B then f is called locally monotone on a set $B\ subset\ of\ N$.

Take an A-definable function f on $D \subseteq M$ and an A-definable equivalence relation E on D. A function f is strictly increasing on D/E whenever for all $a, b \in D$ such that $\neg E(a, b)$ and $a < b$ the following holds $f(a) < f(b)$.

A function f is called to be strictly decreasing on D/E whenever for all $a, b \in D$ such that $\neg E(a, b)$ and $a < b$ the next takes place $f(b) < f(a)$.

Definition 2.4 [47, P. 1511] $RC(\psi(y))$ is designation of the *convexity rank* of $\psi(y)$ for N-definable formula $\psi(y)$ with one free variable where N is a sufficiently saturated model of T, T is "a weakly o-minimal theory" N. C*onvexity rank* of $\psi(y)$ is defined by following way:

1. For infinite $\psi(N)$: $RC(\psi(y)) \geq 1$

2. If $\exists$ an equivalence relation $E(y, z)$ parametrically definable and an infinite family $a_i, i \in \omega$ so that

- $N \vDash \neg E(a_i, a_j)$ for all $i, j \in \omega$ and $i \neq j$,
- $RC(E(y, a_i)) \geq \beta$ and $E(N, a_i)$ is a convex subset of $\psi(N)$ for any i from ω

then $RC(\psi(y)) \geq \beta + 1$

3. If $RC(\psi(y)) \geq \beta$ for all $\beta \leq \delta$ with a limit ordinal δ

then $RC(\psi(y)) \geq \delta$

$RC(\psi(y))$ is called *determinable* if $\exists\, \beta$ such that $RC\big(\psi(y)\big) = \beta$. Otherwise $RC(\psi(y)) = \infty$, i.e. for any β $RC(\psi(y)) \geq \beta$.

Denote by $RC(q)$ the *convexity rank* of a 1-type q. $RC(q)$is the minimal convexity rank of formulas from type q.

Proposition 2.1 [52, P. 387] *If domain of function includes $p(M)$ into an A-definable sort where M is a "weakly o-minimal structure", $A\ is\ subset\ of\ M$, $p\ from\ S_1(A)$ then the function is either locally constant or locally monotone on $p(M)$.*

Theorem 2.1 [56, P. 606] *Let M be a model of $\aleph_0-$categorical theory T which is a weakly o-minimal with finite convexity rank, with $|M|=\omega$. The following holds:*

(i) There exists a finite set $C=\{c_0,\dots,c_n\}\subseteq M$ (or $M\cup\{-\infty,+\infty\}$ when M doesn't have a last or a first element) which consists of all $\emptyset$-definable elements of M except possibly $-\infty$ and $+\infty$ such that $M\vDash c_i<c_j$ for any $n\geq j>i$ and for any $i\in\{1,\dots,n\}$ either there is no element from M between c_{i-1} and c_i or there is a dense linear order without endpoints, i.e. $M\vDash\neg\exists x c_{i-1}<x<c_i$ or $I_i=\{x\in M: M\vDash c_{i-1}<x<c_i\}$ is a dense linear order which doesn't have endpoints, moreover $I_i=\bigcup_{s=1}^{k_i}p_s^i(M)$ for some $k_i\in\omega$ and $p_1^i,\dots,p_{k_i}^i\in S_1(\emptyset)$.

(ii) Every non-algebraic type $p\in S_1(\emptyset)$ has some convexity rank $n_p\geq 1$, where n_p is integer, $RC(p)=n_p$. This means there exists an empty definable equivalence relations: $E_1^p(x,y)$, $E_2^p(x,y)$, ..., $E_{n_p-1}^p(x,y)$ such that

- $p(M)$ is partitioned by $E_{n_p-1}^p$ into infinitely many open and convex $E_{n_p-1}^p$-classes. On these classes the induced order is a dense linear order which doesn't have endpoints.

- for every $i\ from\ \{1,\dots,n_p-2\}$ every E_{i+1}^p-class is partitioned by E_i^p into infinitely many open and convex E_i^p-classes. The E_i^p-subclasses of every E_{i+1}^p-class are linearly ordered dense which doesn't have endpoints.

(iii) For all nonorthogonal, nonalgebraic types $p,q\in S_1(\emptyset)$ with $p\not\perp^w q$ the following holds:

(1) if the definable closure of some realization of p contains some realization of q, that is $dcl(\{\alpha\})\cap q(M)\neq\emptyset$ for some $\alpha\in p(M)$ then there exists a unique $\emptyset$-definable function $f: p(M)\to q(M)$ which is: locally monotone bijection on $p(M)$ whenever $RC(p)=RC(q)$, locally constant on $p(M)$ whenever $RC(p)>RC(q)$ that is on each $E_{n_p-n_q}^p$-class f is constant and on $p(M)/E_{n_p-n_q}^p$ f is locally monotone.

(2) if the definable closure of any realization of p doesn't contain any realization of q, that is $dcl(\{\alpha\})\cap q(M)\neq\emptyset$ for arbitrary $\alpha\in p(M)$ then whenever $RC(p)=RC(q)$ there exists exactly $(2n_p-1)$ (p,q)-splitting formulas $H_1(x,y)$, ..., $H_{2n_p-1}(x,y)$ such that $H_1(\alpha,M)\subset\ldots\subset H_{2n_p-1}(\alpha,M)$ for every $\alpha\in p(M)$. The function $f(x):=sup\ \ H_{n_p}(x,M)$ is locally monotone on $p(M)$

$$H_i(x,y)\equiv\forall t(E_{n_p-i}^p(x,t)\to H_{n_p}(t,y)),\quad 1\leq i\leq n_p-1$$

$$H_j(x,y)\equiv\exists t(E_{j-n_p}^p(x,t)\wedge H_{n_p}(t,y)),\quad n_p+1\leq j\leq 2n_p-1$$

Whenever $RC(p)>RC(q)$ there exist exactly $(2n_q-1)$ (p,q)-splitting formulas $H_1(x,y)$, ..., $H_{2n_q-1}(x,y)$ such that $H_1(\alpha,M)\subset\ldots\subset H_{2n_q-1}(\alpha,M)$ for every $\alpha\in p(M)$. The function $f(x):=sup\ \ H_{n_q}(x,M)$ is locally monotone on $p(M)/$

$E^p_{n_p-n_q}$ and constant on each $E^p_{n_p-n_q}$

$$H_i(x,y) \equiv for\ any\ t(E^p_{n_p-i}(x,t) \to H_{n_q}(t,y)),\ \ n_q - 1 \geq\ i \geq 1$$

$$H_j(x,y) \equiv exist\ t(E^p_{n_p-2n_q+j}(x,t) \wedge H_{n_q}(t,y)),\ \ \ n_q + 1 \leq j \leq 2n_q - 1$$

hence, T admits quantifier elimination to the language

$$\{=,<\} \cup \{c_i : i \leq n\} \cup \{U_s : s \leq r = \sum_{j=1}^{n} k_j\} \cup$$

$$\{f_{i,j} : dcl(\{\alpha\}) \cap p_j(M) \neq \emptyset\ \ for\ \ some\ \ \alpha \in p_i(M), RC(p_i) \geq RC(p_j)\} \cup$$

$$\{H_{i,j}(x,y) : p_i \not\angle p_j, dcl(\{\alpha\}) \cap p_j(M) = \emptyset\ \ for\ \ all\ \ \alpha \in p_i(M), RC(p_i) \geq RC(p_j)\}$$

where for each $s \leq r$, the formula $U_s(x)$ isolates the type p_s. Furthermore there is a weakly o-minimal ω-categorical theory which has finite convexity rank corresponding, as above for every ordering with elements that can be distincted as in (i)-(iii).

Definition 2.5 For a weakly o-minimal structure M, to extent that $A, B \subseteq M$. Assume that that types $p, q \in S_1(A)$ are non-algebraic and M is $|A|^+$-saturated. In case of existance of bijection $f: p(M) \to q(M)$, where f is a function over A, we call type p is not *quite orthogonal* to type q ($p \not\angle^q q$). In the case of coincidence of concepts of weak and quite orthogonality of 1-types we are going to say that an weakly o-minimal theory is *quite o-minimal*.

As soons as in the case of o-minimal theory for every single set A and every single two types over A there is a bijection that is A-definable and strictly monotone between sets of realisations of such types it is clear that any such theory it is also quite o-minimal [86].

Example 2.1 [39, P. 5441] Consider a structure $M = \langle M; <, P_1^1, P_2^1, f^1 \rangle$ that has a property of linear order and the universe M consits of unary predicates P_1 and P_2 interpretations more precisely their disjoint union, with $P_1(M) < P_2(M)$. We identify the P_2 interpretation with the set of rational numbers $\mathbb{Q}$, that is ordered as usual, and P_1 interpretation is $\mathbb{Q} \times \mathbb{Q}$, that is ordered lexicographically. The symbol f defines a partial unary function to extent that D$om\ (f) = P_1(M)$ and R$ange\ (f) = P_2(M)$ interpreted, and the following equality holds $f((n,m)) = n$ for all $(n,m) \in \mathbb{Q} \times \mathbb{Q}$.

It can be verified that $Th(M)$ has a weakly o-minimal theory. Consider the following types $p(x) := \{P_1\}$, $q(x) := \{P_2\}$. It is clear that types $p, q \in S_1(\emptyset)$ and $p \perp^q q$ with $p \not\perp^w q$, so we can conclude that $Th(M)$ is not quite o-minimal. Also note that $RC(p) = 2, RC(q) = 1$.

Example 2.2 Consider a structure $\mathrm{N} = \langle N, <, H_1^1, H_2^1, E_1^2, E_2^2, f^1 \rangle$ that is linearly ordered and to extent that N is the ditinct union of P_1 and P_2 unary predicates realisations, to extent that realisations ordered as follows $H_1(\mathrm{N}) < H_2(\mathrm{N})$. Thus realisations of predicates H_1 and H_2 is similar to the $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. Two binary predicates $E_1(t, z)$ and $E_2(t, z)$ are interpreted as equivalence relations on $H_1(\mathrm{N})$ and $H_2(\mathrm{N})$ respectively. Relations of quvalence for every $t = (l_1, s_1), z = (l_2, s_2)\ from\ \mathbb{Q} \times \mathbb{Q}$, are defined by next way:

For $E_i(t, z)\ necessary\ and\ sufficient\ condition\ is\ l_1 = l_2,\ \ where\ \ i = 1{,}2.$

A symbol g is interpreted by a partial unary function with D$om\,(g) = H_1(\mathrm{N})$ and R$ange\,(g) = H_2(\mathrm{N})$. It is determined as follows: $g((l, s)) = (l, -s)$ for every $(l, s)\ from\ \mathbb{Q} \times \mathbb{Q}$.

It can be understood that $H_1(\mathrm{N})$ and $H_2(\mathrm{N})$ are partitioned into an infinite number of infinite convex classes by $\emptyset$-definable equivalence relations $E_1(t, z)$ and $E_2(t, z)$ respectively. We are saying that function g is strictly decreasing on any $E_1(b, \mathrm{N})$, where $b\ from$ (N), and on $H_1(\mathrm{N})/E_1$ function g is strictly increasing. It is easy to prove, that T$h(\mathrm{N})$ is “a quite o-minimal theory”. A convex set defined by set $E_1(b, N)$ is not an interval in N. For this reason the theory T$h(\mathrm{N})$ is not o-minimal. Note also that $RC(H_1(t)) = RC(H_2(t)) = 2$.

Quite o-minimal theories are a subclass of weakly o-minimal theories which has many o-minimal theories properties. In [53, P. 45] it were described examples of quite o-minimal countably categorical theories. Their binarity follows from this description (o-minimal countably categorical theories has a similar result).

Theorem 2.2 *Suppose T is a countably categorical theory quite o-minimal, and $N \vDash T$ where $|N| = \aleph_0$. Thereat the following holds* [52, P. 390 ; 53, P. 48]:

(i) there is a finite set $D = \{d_0, \dots, d_n\}\ is\ subset\ of\ N(N \cup\ \{-\infty,\ +\infty\}$, whensoever N hasn’t first or last element), which consists of all empty elements definable in N (with eventual exceptions for $-\infty$, $+\infty$), such that $N \vDash d_i < d_j$ for any $i < j \leq n$ and for all $j\ from$ $\{1, \dots, n\}$ either $N \vDash \neg(exists\ x) d_{j-1} < x < d_j$ or $I_j = \{\ x\ from\ N : N \vDash d_{j-1} < x < d_j\ \}$ is “a dense linear order without endpoints” and there exists $k_j\ from\ \omega$ and $q_1^j, \dots, q_{k_j}^j\ from\ S_1(\emptyset)$ that $I_j = \bigcup_{s=1}^{k_j} q_s^j(N)$;

(ii) let $q\ from\ S_1(\emptyset)$ is any non algebraic one-types, then there exists $n_q\ from\ \omega$ such that $RC(q) = n_q$, so that there is empty definable equivalence relations $E_1^q(x, y)$, $E_2^q(x, y)$, ..., $E_{n_q-1}^q(x, y)$ that the following holds

- an induced order on the classes is “a dense linear order without endpoints” because $E^q_{n_q-1}$ partitions $q(N)$ into an infinite number of convex and open classes $E^q_{n_q-1}$

- for any $i\,from\,\{1,\dots,n_q-2\}$ E^q_i splits up every E^q_{i+1}-class into an infinite number of convex and open classes E^q_i, so that the set of subclasses E^q_i of every class E^q_{i+1} is “densely linearly ordered without endpoints”

(iii) a relation of equivalence $\varepsilon \subseteq (\{s: 1 \le s \le k\})^2$ exists, such that any non-algebraic 1-types over empty set arbitrary enumeration is $\{q_s \mid s \le k < \omega\}$, and for each $(i,j) \in \varepsilon$ there is a unique locally monotone empty definable bijection $f_{i,j}: q_i(N) \to q_j(N)$ that $RC(q_i) = RC(q_j)$, $f_{i,i} = id_{p_i(M)}$ and $f_{j,l} \circ f_{i,j} = f_{i,l}$ for all (i,j), $(j,l) \in \varepsilon$ to such an extent that T accepts quantifier exception up to the language $\{=,<\} \cup \{d_i: i \le n\} \cup \{U_s(x): s \le k\} \cup \{E^{q_s}_l(x,y): s \le k, l \le n_{q_s}\} \cup \{f_{i,j}: (i,j) \in \varepsilon\}$, such that $U_s(x)$ insulates the q_s type for each $s \le k$.

Moreover, any ordering of chosen elements as mentioned in (i)-(ii) and any appropriate relation of equivalence ε as mentioned in (iii) meets to a quite o-minimal countably categorical theory as set out above.

Definition 2.6 [87] Suppose N is a weakly o-minimal structure, $B\ is\ subset\ of\ N$, N is $|B|^+$-satiated, $q\ from\ S_1(B)$ is non-algebraic.

(1) It is called that an B-formula $F(x,y)$ is *q-preserving* (or *q-stable*) if there exists the type $\beta, \gamma_1, \gamma_2\ from\ q(N)$ realisations such that

$$[F(N,\beta)\backslash\{\beta\}] \cap q(N) \neq \emptyset$$

and $\gamma_1 < F(N,\beta) \cap q(N) < \gamma_2$.

(2) It is called that a formula $F(x,y)$ is *convex to the right (left)* if there is q-stable and there exists $\beta\ from\ p(N)$ that $F(N,\beta) \cap q(N)$ is convex and the β is in $F(N,\beta)$ and it is the left (right) endpoint of $F(N,\beta) \cap q(N)$.

Definition 2.7 [54, P. 31] A formula $F(t,z)$ is called to be is *equivalence-generating* if there is q-preserving convex to the left (right) and for each realisations $\beta, \gamma\ from\ q(N)$ such that $N \vDash F(\gamma,\beta)$, we have the following:

$$N \vDash \text{for any } t(t \le \gamma \to (F(t,\beta) \leftrightarrow F(t,\gamma)))$$

$$(resp.\ \ N \vDash \text{for any } t(t \ge \gamma \to (F(t,\beta) \leftrightarrow F(t,\gamma))))$$

Lemma 2.2 [54, P. 33] *Suppose N is a weakly o-minimal structure, $B\ is\ subset\ of\ N$, N is $|B|^+$-satiated, non-algebraic type $q\ from\ S_1(B)$, $F(t,y)$ is a q-preserving convex to the right (left) formula. If $F(t,\ z)$ is not equivalence-generating there exists realisations $\beta, \gamma\ from\ q(N)$ such that*

$$N \vDash F(\gamma,\beta) \wedge exists\ t(\neg F(t,\beta) \wedge F(t,\gamma))$$

Lemma 2.3 [54, P. 34] *Suppose N is "a weakly o-minimal structure", $B\ is\ subset\ of\ N$, N is $|B|^+$-saturated, non-algebraic type $q \in S_1(B)$, convex to the right (left) q-preserving formula $F(t,z)$. In this case a formula $F'(t,z) := exists\ x(F(x,z) \wedge F(t,x))$ is also convex to the right (left) q-preserving.*

Lemma 2.4 [54, P. 36] *Suppose N is a "weakly o-minimal structure", $B\ is\ subset\ of\ N$, N is $|B|^+$-saturated, $F(t,z)$ is a equivalence-generating formula q-preserving convex to the right (left). Then*:

1) $G(t,z) := F(z,t)$ is a formula q-preserving convex to the left (right) and $G(t,z)$ is also equivalence-generating.

2) $E(t,z) := F(t,z) \vee F(z,t)$ is a relation of equivalence that splits up $q(N)$ into infinitely many infinite convex classes.

Proposition 2.2 [54, P. 37] *Suppose T is a countably categorical "weakly o-minimal theory", N is a model of theory T, $B\ is\ subset\ of\ N$, non-algebraic one-type $q\ from\ S_1(B)$. Then any convex formula q-preserving to the right (left) is equivalence-generating.*

2.1 Unary expansions

Suppose M is a model of "weakly o-minimal theory". For some formula $\varphi(x)$ denote the set $\{b \in M | \forall x(\varphi(x) \to b < x)\}$ as $\varphi(M)^-$. Denote $\varphi(M)^+$ the set $\{b \in M | \forall x(\varphi(x) \to x < b)\}$.

Suppose M is a model of "weakly o-minimal theory", $A \subset M$, $p(x) \in S_1(A)$. Denote the set $\{b \in M | \forall x(p(x) \to b < x)\}$ as $p(M)^-$. Denote $p(M)^+$ the set $\{b \in M | \forall x(p(x) \to x < b)\}$.

For any weakly o-minimal countably categorical structure $M := \langle M, \Sigma \rangle$ any expansion by a new convex unary predicate $U(x)$ preserves weak o-minimality, that is $M' := \langle M, \Sigma, U^1 \rangle$ has $T' := Th(M')$ "weakly o-minimal theory" [8, P. 1382]. It is worth noting that an expansion using an unary predicate $U(x)$ with interpretation which is a finite quantity of convex sets in M, say m, is equivalent to the expansion by a finite number of unary convex predicates $U_i(x)$, for $1 \le i \le m$, because all these convex sets are $\emptyset$-definable. As M is countably categorical there exists only a finite number of non-algebraic 1-types over $\emptyset$. Call them $p_1, p_2, \ldots, p_s$. Without loss of generality assume:

$$p_1(M) < p_2(M) < \ldots < p_s(M)$$

Let $U(M)$ lie across p_1, p_2, p_3, $p_2(M) \subset U(M)$, so that there exists $\alpha_1, \alpha_2 \in p_1(M)$ and $\beta_1, \beta_2 \in p_3(M)$ such that

$$M' \vDash \alpha_1 < \alpha_2 \wedge \neg U(\alpha_1) \wedge U(\alpha_2) \wedge \beta_1 < \beta_2 \wedge U(\beta_1) \wedge \neg U(\beta_2)$$

Then the introduction of the unary predicate $U(x)$ is equivalent to the introduction of two convex unary predicates $U_1(x) := p_1(x) \wedge \neg U(x)$ and $U_2(x) := p_3(x) \wedge U(x)$. Hence we will consider a unary predicate $U(x)$ such that $U(M) \subset p(M)$ and $U(M)^- = p(M)^-$ for some non-algebraic one-type $p\ from\ S_1(\emptyset)$, it means that there is $\alpha\ from\ p(M)$ that $\alpha > U(M)$.

If the right boundary of $U(x)$ is determined by some $b \in M$, then this expansion is definable, non essential, and equivalent to extension of M by one constant. Obviously all the properties of initial model is preserved in this case and T' keeps being countably categorical. Therefore further we consider the case, that the right boundary of $U(x)$ doesn't lie in M and hence determine irrational cut in M.

Suppose an $E(x, y)$ is a relation of equivalence definable over empty set, that splits out $p(M)$ into infinite number of classes eash of them is infinite and convex. A predicate $U(x)$ is irrational with respect to the E-classes whenever the following holds:

(1) for each $\alpha\ from\ p(M)$ for which $U(\alpha)$ takes place, there exists a realisation $\beta\ from\ p(M)$ where

$$M' \vDash \alpha < \beta \wedge \neg E(\alpha, \beta) \wedge U(\beta)$$

(2) for each $\alpha\ from\ p(M)$ for which $\neg U(\alpha)$ takes place, there exists a realisation $\beta\ from\ p(M)$ where

$$M' \vDash \beta < \alpha \wedge \neg E(\alpha, \beta) \wedge \neg U(\beta)$$

Let's see the next sample.

Example 2.3 Consider a linearly ordered structure $M := \langle \mathbb{Q}, <, U^1 \rangle$, where $U(x) := \{b \in Q : b < \sqrt{2}\}$ is a unary convex predicate in M. Replace each element $a \in M$ by a copy of the set of rationals $\mathbb{Q}$ and define $E(x, y)$ new binary relation by next way: for any $a_1 = (m_1, n_1), a_2 = (m_2, n_2) \in \mathbb{Q} \times \mathbb{Q}$ they are in the similar equivalence class if their first coordinates match, that is

$$E(a_1, a_2) \Leftrightarrow m_1 = m_2$$

We obtain the structure $M' := \langle \mathbb{Q} \times \mathbb{Q}, <, U^1, E^2 \rangle$, which is splited out into infinite number of convex classes by the equivalence relation $E(x, y)$, so that the order induced on the E-classes is a dense linear order which doesn't include endpoints.

M' is a weakly o-minimal countably categorical structure. The predicate $U(x)$ is irrational with respect to the E-classes.

$U(x)$ is called quasirational to the right (left) with respect to the E-classes $\Leftrightarrow$

there exists $\alpha\ from\ p(M)$ and there is an E-class $E(\alpha, M)$ that

$$U(M)^+ = E(\alpha, M)^+ \quad (U(M) = p(M) \cap E(\alpha, M)^-)$$

Lemma 2.5 *Let type $p \in S_1(\emptyset)$ be non-algebraic, and relation of equivalence $E(x, y)$ definable over empty set splits out $p(M)$ into infinite number of classes which are convex and infinite. In the case when with respect to the E-classes $U(x)$ is quasirational to the right (left) there is an empty definable quasirational to the left (right) convex formula $U'(x)$ with respect to the E-classes.*

Proof: It is possible to expect that $U(x)$ is quasirational to right point with respect to the E-classes, as proof for the case of quasirational to the left is similar. Hence there exists $\alpha \in p(M)$ with $U(M)^+ = E(\alpha, M)^+$. Consider the formula

$$R(x) := \exists z(\neg E(x, z) \wedge x < z \wedge U(x) \wedge \neg U(z) \wedge \forall t_1(E(x, t_1) \to U(t_1)) \wedge$$
$$\wedge \forall t_2(x < t_2 < z \wedge \neg E(x, t_2) \to \neg U(t_2)))$$

So $U'(x) := U(x) \wedge \neg R(x)$ is quasirational to the right (left) with respect to the E-classes empty definable convex formula.

Theorem 2.3 [58, P. 207] *Suppose for some $k < \omega$ M is a model of a countably categorical "weakly o-minimal theory" of convexity k-rank, and M' is the expansion of M by a finite family $\{U_i : i < m\}$ of convex unary predicates , where $m < \omega$. Then the theory $Th(M')$ also is a countably categorical "weakly o-minimal theory" of convexity rank k.*

In order to prove Theorem 2.3 we will require several lemmas. We assume for now that T is a weakly o-minimal countably categorical theory which has convexity of finite rank, and $M \models T$.

Lemma 2.6 *Given non-algebraic type $p \in S_1(\emptyset)$ with $RC(p) = n$, the predicate $U(x)$ partitions a number of type p realizations into s $\emptyset$-definable 1-indiscernible convex sets, where $2 \le s \le 2n$.*

Proof: Since T is a countably categorical theory, the type p is isolated, hence there exists an $\emptyset$-definable isolating formula $P(x)$. As $RC(p) = n$, there is relations of equivalence definable by parametrs $E_1(t, z), E_2(t, z), \ldots, E_{n-1}(t, z)$ and the relations split out $p(M)$ into infinite number of classes which are infinite and convex such that holds

$$E_{n-1}(\alpha, M) \supset \cdots \supset E_2(\alpha, M) \supset E_1(\alpha, M)$$

for some $\alpha\ from\ p(M)$.

Relation of equivalence $E_{n-1}(t, z)$ splits out $p(M)$ into ordered by type $\mathbb{Q}$ infinite number classes which are infinite and convex. Every class $E_{i+1}$is splited out into infinite number of E_i-subclasses which are convex, ordered by type $\mathbb{Q}$, $(1 \le i \le$

$n-2$). Each E_1-class is 2-indiscernible over $\emptyset$. Therefore it is sufficient to study the mutual location of E_i-classes and the predicate $U(x)$, where $1 \le i \le n-1$.

Consider the formulas:

$$E_0(x,y) := x = y, \quad E_n(x,y) := P(x) \wedge P(y)$$

$$\theta_i := \exists z_1, z_2: (E_i(y,z_1) \wedge E_i(z_1,z_2) \wedge U(z_1) \wedge \neg U(z_2)), \quad 1 \le i \le n-1$$

$$R_i(y) \coloneqq \exists z(E_{i+1}(y,z) \wedge \neg E_i(y,z) \wedge y < z \wedge U(y) \wedge \neg U(z) \wedge$$
$$\wedge \forall t_1(E_i(y,t_1) \to U(t_1)) \wedge \forall t_2(y < t_2 < z \wedge \neg E_i(y,t_2) \to \neg U(t_2))), \quad 1 \le i \le n-1,$$

$$L_i(y) := \exists z(E_{i+1}(y,z) \wedge \neg E_i(y,z) \wedge y > z \wedge \neg U(y) \wedge U(z) \wedge$$
$$\wedge \forall t_1(E_i(y,t_1) \to \neg U(t_1)) \wedge \forall t_2(z < t_2 < y \wedge \neg E_i(y,t_2) \to U(t_2))), \quad 1 \le i \le n-1.$$

Case 1. Predicate $P(x)$ defines an irrational cut with respect to the E_{n-1}-classes. For this case $\theta_i(x)$ doesn't have realzations in M for all $1 \le i \le n-1$ and $P(x)$ is divided into two convex formulas: $P(x) \wedge U(x)$ and $P(x) \wedge \neg U(x)$.

Case 2a. For some $2 \le i \le n-1$ each class E_i is divided by the predicate $U(x)$ and $U(x)$ is irrational with respect to the E_{i-1}-classes. In this case $P(x)$ is divided into the $2(n-i+1)$ formulas:

$$U_j^l(x) := P(x) \wedge \forall y(\theta_i(y) \to x < y \wedge \neg E_{i+j-2}(x,y) \wedge E_{i+j-1}(x,y)),$$

$$U_1^l(x) := P(x) \wedge \forall y(\theta_i(y) \to E_i(x,y) \wedge U(x)),$$

$$U_1^r(x) := P(x) \wedge \forall y(\theta_i(y) \to E_i(x,y) \wedge \neg U(x))$$

$$U_j^r(x) := P(x) \wedge \forall y(\theta_i(y) \to x > y \wedge \neg E_{i+j-2}(x,y) \wedge E_{i+j-1}(x,y))$$

where $2 \le j \le n-i+1$.

Case 2b. Some E_1-class is divided by $U(x)$. In this case $P(x)$ is divided into the $2n$ formulas:

$$U_j^l(x) := P(x) \wedge \forall y(\theta_i(y) \to x < y \wedge \neg E_{i+j-2}(x,y) \wedge E_{i+j-1}(x,y)),$$

$$U_j^r(x) := P(x) \wedge \forall y(\theta_1(y) \to x > y \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y)), \quad 1 \le j \le n-1$$

$$U_0^l(x) := P(x) \wedge \theta_1(x) \wedge U(x), \quad U_0^r(x) := P(x) \wedge \theta_1(x) \wedge \neg U(x)$$

$$U_j^r(x) := P(x) \wedge \forall y(\theta_1(y) \to x > y \wedge \neg E_j(x,y) \wedge E_{j+1}(x,y)), \quad 1 \le j \le n-1$$

Case 3. For some $2 \leq i \leq n$ each E_i-class is divided by the predicate $U(x)$ and $U(x)$ is quasirational to the right with respect to the E_{i-1}-classes. In this case $P(x)$ divided into the $2(n-i)+3$ formulas:

$$U_j^l(x) := P(x) \wedge \forall y(\theta_i(y) \rightarrow x < y \wedge \neg E_{n-j}(x,y) \wedge E_{n-j+1}(x,y)),$$

$$U_1^l(x) := P(x) \wedge \forall y(R_{i-1}(y) \rightarrow x < y \wedge E_i(x,y)), \quad U_0^l(x) := P(x) \wedge R_{i-1}(x),$$

$$U_1^r(x) := P(x) \wedge \forall y(R_{i-1}(y) \rightarrow x > y \wedge E_i(x,y))$$

$$U_j^r(x) := P(x) \wedge \forall y(\theta_i(y) \rightarrow x > y \wedge \neg E_{n-j}(x,y) \wedge E_{n-j+1}(x,y))$$

where $2 \leq j \leq n-i+1$.

Case 4. For some $2 \leq i \leq n$ each class E_i is divided by the predicate $U(x)$ and $U(x)$ is quasirational to the left with respect to the E_{i-1}-classes. In this case $P(x)$ is divided into the $2(n-i)+3$ formulas:

$$U_j^l(x) := P(x) \wedge \forall y(\theta_i(y) \rightarrow x < y \wedge \neg E_{n-j}(x,y) \wedge E_{n-j+1}(x,y)),$$

$$U_1^l(x) := P(x) \wedge \forall y(L_{i-1}(y) \rightarrow x < y \wedge E_i(x,y)), \quad U_0^l(x) := P(x) \wedge L_{i-1}(x),$$

$$U_1^r(x) := P(x) \wedge \forall y(L_{i-1}(y) \rightarrow x > y \wedge E_i(x,y))$$

$$U_j^r(x) := P(x) \wedge \forall y(\theta_i(y) \rightarrow x > y \wedge \neg E_{n-j}(x,y) \wedge E_{n-j+1}(x,y))$$

where $2 \leq j \leq n-i+1$.

Lemma 2.7 *Let $p, q \in S_1(\emptyset)$ be two non-algebraic one-types over the empty set, such that $p \not\perp^w q$ and $RC(p) = RC(q) = n$. If $p(M)$ is partitioned into s $\emptyset$-definable convex sets, then $q(M)$ is also partitioned into s $\emptyset$-definable convex sets.*

Proof: Let be an $P(x)$ isolating formula of p. As $RC(p) = RC(q)$, there exists a (p,q)-splitting formula $R(x,y)$ such that the function $f(x) := sup \ \ R(x,M)$ is locally monotone on $p(M)$ [56, P. 606]. We list, in each of the following cases, the convex formulas partitioning q.

Case 1. $U(x)$ is irrational with respect to the E_{n-1}-classes. Lemma 2.6 shows that $P(x)$ is divided into the two formulas

$$U_1^l(x) := P(x) \wedge U(x)$$

and

$$U_1^r(x) := P(x) \wedge \neg U(x).$$

If f increases strictly on $p(M)/E_{n-1}$ then

$$Q_1^l(x) := \exists t(U_1^l(t) \wedge R(t,x)), \quad Q_1^r(x) := \exists t(U_1^r(t) \wedge R(t,x)) \wedge \neg Q_1^l(x)$$

In case if f is strictly decreasing on $p(M)/E_{n-1}$ then

$$Q_1^l(x) := \exists t(U_1^r(t) \wedge R(t,x)), \quad Q_1^r(x) := \exists t(U_1^l(t) \wedge R(t,x)) \wedge \neg Q_1^l(x)$$

Case 2a. For some $2 \le i \le n-1$ some E_i-class is divided by predicate $U(x)$ and $U(x)$ is irrational with respect to the E_{i-1}-classes.

In case when f is strictly increasing on $p(M)/E_{n-1}$

$$Q_{n-i+1}^l(x) := \exists t(U_{n-i+1}^l(t) \wedge R(t,x))$$

$$Q_{n-i+1}^r(x) := \exists t\big(U_{n-i+1}^r(t) \wedge R(t,x)\big) \wedge \neg Q_{n-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \\ \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n-i}^r(x).$$

If f is strictly decreasing on $p(M)/E_{n-1}$

$$Q_{n-i+1}^l(x) := \exists t(U_{n-i+1}^r(t) \wedge R(t,x))$$

$$Q_{n-i+1}^r(x) := \exists t\left(U_{n-i+1}^l(t) \wedge R(t,x)\right) \wedge \neg Q_{n-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \\ \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n-i}^r(x).$$

Case 2b. Some E_1-class is divided by predicate $U(x)$. If f is strictly increasing on $p(M)/E_{n-1}$ then

$$Q_{n-1}^l(x) := \exists t(U_{n-1}^l(t) \wedge R(t,x))$$

$$Q_{n-1}^r(x) := \exists t\big(U_{n-1}^r(t) \wedge R(t,x)\big) \wedge \neg Q_{n-1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_0^l(x) \wedge \\ \wedge \neg Q_0^r(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n-2}^r(x).$$

If f is strictly decreasing on $p(M)/E_{n-1}$ then

$$Q_{n-1}^l(x) := \exists t(U_{n-1}^r(t) \wedge R(t,x))$$

$$Q_{n-1}^r(x) := \exists t\left(U_{n-1}^l(t) \wedge R(t,x)\right) \wedge \neg Q_{n-1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_0^l(x) \wedge \\ \wedge \neg Q_0^r(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n-2}^r(x).$$

Furthermore, if f is strictly increasing on $E_{m+1}(\alpha, M)/E_m$ for some $\alpha \in p(M)$, where $0 \le m \le n-2$, then

$$Q_m^l(x) := \exists t(U_m^l(t) \wedge R(t,x)) \wedge \neg Q_{n-1}^l(x) \wedge \ldots \wedge \neg Q_{m+1}^l(x)$$

$$Q_{n-1}^r(x) := \exists t\big(U_m^r(t) \wedge R(t,x)\big) \wedge \neg Q_{n-1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_0^l(x) \wedge$$
$$\wedge \neg Q_0^r(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{m-1}^r(x).$$

In the case when f is strictly decreasing on $E_{m+1}(a,M)/E_m$ for some $\alpha \in p(M)$, where $0 \leq m \leq n-2$

$$Q_m^l(x) := \exists t(U_m^r(t) \wedge R(t,x)) \wedge \neg Q_{n-1}^l(x) \wedge \ldots \wedge \neg Q_{m+1}^l(x)$$

$$Q_{n-1}^r(x) := \exists t\left(U_m^l(t) \wedge R(t,x)\right) \wedge \neg Q_{n-1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_0^l(x) \wedge$$
$$\wedge \neg Q_0^r(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{m-1}^r(x).$$

Case 3. For some $2 \leq i \leq n-1$ some E_i-class is divided by the predicate $U(x)$ and $U(x)$ is quasirational to the right with respect to the E_{i-1}-classes.

$$Q_{n-i+1}^l(x) := \exists t(U_{n-i+1}^l(t) \wedge R(t,x)),$$

$$Q_0^r(x) := \exists t(U_0^l(t) \wedge R(t,x)) \wedge \neg Q_{n-i+1}^l(x) \wedge \ldots \wedge \neg Q_0^l(x).$$

$$Q_{n-i+1}^r(x) := \exists t\big(U_{n-i+1}^r(t) \wedge R(t,x)\big) \wedge \neg Q_{n-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_0^l(x) \wedge$$
$$\wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n-i}^r(x).$$

Case 4. For some $2 \leq i \leq n$ some E_i-class is divided by the predicate $U(x)$ and $U(x)$ is quasirational to the right with respect to the E_{i-1}-classes.

$$Q_{n-i+1}^l(x) := \exists t(U_{n-i+1}^l(t) \wedge R(t,x)),$$

$$Q_0^r(x) := \exists t(U_0^r(t) \wedge R(t,x)) \wedge \neg Q_{n-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x).$$

$$Q_{n-i+1}^r(x) := \exists t\big(U_{n-i+1}^r(t) \wedge R(t,x)\big) \wedge \neg Q_{n-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge$$
$$\wedge \neg Q_0^r(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n-i}^r(x).$$

Lemma 2.8 *Let $p, q \in S_1(\emptyset)$ be two non-algebraic one-types over the empty set, such that $p \not\perp^w q$ and $RC(p) > RC(q)$. If $p(M)$ is partitioned into s $\emptyset$-definable convex sets, then $q(M)$ is partitioned into l $\emptyset$-definable convex sets, where $2 \leq l \leq s$.*

Proof: Let $RC(p) = n_p$ and $RC(q) = n_q$. There exists an (p,q)-splitting formula $R(x,y)$ such that $f(x) := sup \ \ R(x,M)$ is constant on each E_j^p-class [56, P. 606], furthermore $E_j^p(x,y)$ is the greatest equivalence on $p(M)$ with this property and f is locally monotone on $p(M)/E_j^p$ where $j = n_p - n_q$

Case 1. $U(x)$ is irrational with respect to the $E_{n_p-1}^p$-classes. The existence of a (p,q)-splitting formula implies that $P(x)$ is divided into two formulas $U_1^l(x)$ and $U_1^r(x)$. An f is strictly monotone on $p(M)/E_{n_p-1}^p$. Thus if f strictly increasing on $p(M)/E_{n_p-1}^p$ then

$$Q_1^l(x) := \exists y(U_1^l(y) \wedge R(y,x)), \quad Q_1^r(x) := \exists y(U_1^r(y) \wedge R(y,x)) \wedge \neg Q_1^l(x)$$

If f is strictly decreasing on $p(M)/E_{n_p-1}^p$ then

$$Q_1^l(x) := \exists y(U_1^r(y) \wedge R(y,x)), \quad Q_1^r(x) := \exists y(U_1^l(y) \wedge R(y,x)) \wedge \neg Q_1^l(x)$$

Case 2a. Some E_i^p-class is divided by the predicate $U(x)$ for some $2 \le i \le n_{p-1}$ and $U(x)$ is irrational with respect to the E_{i-1}^p-classes. In this event $p(M)$ is partitioned by $U(x)$ into $2(n_p - i + 1)$ $\emptyset$-definable convex sets. In the case when $i \le j$ the formulas $Q_m^l(x)$ and $Q_m^r(x)$, for $j - i + 2 \le m \le n_p - i + 1$, are defined as in Lemma 2.6 case 2a. If $dcl(\{\alpha\}) \cap q(M) \ne \emptyset$, where $\alpha \in p(M)$, then additionally we define the formula

$$Q_{j-i+1}^c(x) := \exists y \left[\bigwedge_{t=1}^{j-i+1} (U_t^l(y) \vee U_t^r(y)) \wedge f(y) = x \right]$$

It's clear that $Q_{j-i+1}^c(M) = \{\beta\}$ for some $\beta \in q(M)$. Formula is taken off if $dcl(\{\alpha\}) \cap q(M) = \emptyset$ for every $\alpha\ from\ p(M)$. Therefore number of formulas are at most $2(n_p - j) + 1$.

When $j < i$ the formulas $Q_m^r(x)$ and $Q_m^l(x)$, for $n_p - i + 1 \ge m \ge 1$, are defined as in Lemma 2.6 case 2a, and there are at most $2(n_p - i + 1)$ of them.

Case 2b. Some E_1^p-class is divided by $U(x)$. In this case $i \le j$ and we obtain $2(n_p - j)$ formulas: $Q_m^r(x)$ and $Q_m^l(x)$, where $j \le m \le n_p - 1$, if $dcl(\{\alpha\}) \cap q(M) = \emptyset$ for each $\alpha \in p(M)$. Otherwise the formula $Q_{j-1}^c(x)$ is added.

Case 3. Some E_i^p-class is divided by $U(x)$ for some $2 \le i \le n_p$ and $U(x)$ is quasirational to the right with respect to the E_{i-1}^p-classes. In the case when $i \le j$ we obtain $2(n_p - j) + 1$ formulas: $Q_m^r(x)$ and $Q_m^l(x)$, where $j - i + 2 \le m \le n_p - i + 1$. In the case when $i > j$ we obtain $2(n_p - i) + 3$ formulas: $Q_m^r(x)$ where $1 \le m \le n_p - i + 1$ and $Q_m^l(x)$ for m from interval: $n_p + 1 \ge m \ge 0$.

Lemma 2.9 *Suppose $p, q\ from\ S_1(\emptyset)$ are two one-types non-algebraic which are over the empty set, such that $RC(q) > RC(p)$ and $p \not\perp^w q$. If $p(M)$ is partitioned* *into s $\emptyset$-definable convex sets, then $q(M)$ is partitioned into l $\emptyset$-definable convex sets, where* $2 \le l \le s$.

Proof: There exists a (q,p)-splitting formula $R(x,y)$ such that the function $f(x) := sup \quad R(x,M)$ is constant for all E_j^q-classes [56, P. 606], furthermore $E_j^q(x,y)$ is the greatest equivalence on $q(M)$ with this property and f is locally monotone on $p(M)/E_j^q$ where $j = n_q - n_p$

Case 1. $U(x)$ is irrational with respect to the $E_{n_p-1}^p$-classes. Existence of a (q,p)-splitting formula implies that $P(x)$ is divided into two formulas $U_1^l(x)$ and $U_1^r(x)$. An f is strictly monotone on $q(M)/E_{n_q-1}^q$. Thus if f strictly increasing on $q(M)/E_{n_q-1}^q$ then

$$Q_1^l(x) := \forall x_1 \forall y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \rightarrow U_1^l(y))$$

$$Q_1^r(x) := \exists x_1 \exists y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \wedge U_1^r(y)) \wedge \neg Q_1^l(x)$$

If f is strictly decreasing on $q(M)/E_{n_q-1}^q$ then

$$Q_1^l(x) := \forall x_1 \forall y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \rightarrow U_1^r(y))$$

$$Q_1^r(x) := \exists x_1 \exists y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \wedge U_1^l(y)) \wedge \neg Q_1^l(x)$$

Case 2a. E_i^p-class is divided by the predicate $U(x)$ for some $2 \le i \le n_{p-1}$ and $U(x)$ is irrational with respect to the E_{i-1}^p-classes. In this event $p(M)$ is partitioned by $U(x)$ into $2(n_p - i + 1)$ $\emptyset$-definable convex sets. In the case when on $q(M)/E_{n_q-1}^q$ - f is strictly increasing holds

$$Q_1^r(x) := for\ some\ x_1\ for\ some\ y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \wedge U_{n_p-i+1}^r(y)) \wedge \\ \wedge \neg Q_{n_p-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n_p-i}^l(x)$$

$$Q_{n_p-i+1}^l(x) := for\ any\ x_1 for\ any\ y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \rightarrow U_{n_p-i+1}^l(y))$$

In the case when on $q(M)/E_{n_q-1}^q$ - f is strictly decreasing holds

$$Q_1^r(x) := for\ some\ x_1\ for\ some\ y (E_{n_q-1}^q(x,x_1) \wedge R(x_1,y) \wedge U_{n_p-i+1}^l(y)) \wedge \\ \wedge \neg Q_{n_p-i+1}^l(x) \wedge \ldots \wedge \neg Q_1^l(x) \wedge \neg Q_1^r(x) \wedge \ldots \wedge \neg Q_{n_p-i}^l(x)$$

$$Q^l_{n_p-i+1}(x) := for\ any\ x_1 for\ any\ y(E^q_{n_q-1}(x,x_1) \wedge R(x_1,y) \to U^r_{n_p-i+1}(y))$$

In the case when f is strictly increasing on $E^q_{j+m+i-2}(\alpha,M)/E^q_{j+m+i-1}$ for some $\alpha \in q(M)$, where $1 \le m \le n_p - i$

$$Q^l_m(x) := \forall x_1 \forall y \left(E^q_{j+m+i-2}(x,x_1) \wedge R(x_1,y) \to U^l_m(y)\right) \wedge \\ \wedge \neg Q^l_{n_p-i+1}(x) \wedge \ldots \wedge \neg Q^l_{m+1}(x)$$

$$Q^r_m(x) := \exists x_1 \exists y \left(E^q_{j+m+i-2}(x,x_1) \wedge R(x_1,y) \wedge U^r_m(y)\right) \wedge \\ \wedge \neg Q^l_{n_p-i+1}(x) \wedge \ldots \wedge \neg Q^l_1(x) \wedge \neg Q^r_1(x) \wedge \ldots \wedge \neg Q^r_{m-1}(x)$$

In the case when f is strictly decreasing on $E^q_{j+m+i-2}(\alpha,M)/E^q_{j+m+i-1}$ for some $\alpha \in q(M)$, where $1 \le m \le n_p - i$

$$Q^l_m(x) := \forall x_1 \forall y \left(E^q_{j+m+i-2}(x,x_1) \wedge R(x_1,y) \to U^r_m(y)\right) \wedge \\ \wedge \neg Q^l_{n_p-i+1}(x) \wedge \ldots \wedge \neg Q^l_{m+1}(x)$$

$$Q^r_m(x) := \exists x_1 \exists y \left(E^q_{j+m+i-2}(x,x_1) \wedge R(x_1,y) \wedge U^l_m(y)\right) \wedge \\ \wedge \neg Q^l_{n_p-i+1}(x) \wedge \ldots \wedge \neg Q^l_1(x) \wedge \neg Q^r_1(x) \wedge \ldots \wedge \neg Q^r_{m-1}(x)$$

Hence we obtain $2(n_p - i + 1)$ number of formulas.

Case 2b. Some E^p_1-class is divided by $U(x)$. In this case we obtain $2n_p$ formulas: $Q^r_m(x)$ and $Q^l_m(x)$, for $0 \le m \le n_p - 1$.

Case 3. Some E^p_i-class is divided by $U(x)$ for some $2 \le i \le n_p$ and with respect to the E^p_{i-1}-classes $U(x)$ is quasirational to the right. $2(n_p - i) + 3$ formulas are obtained in this case: $Q^r_m(x)$ where $1 \le m \le n_p - i + 1$ and $Q^l_m(x)$, where $0 \le m \le n_p + 1$.

Now we can prove Theorem 2.3.

Proof: Let $RC(p) = n_p$. Based on Lemma 2.6 a number of implementation of $p(M')$ is partitioned by $U(x)$ into s $\emptyset$-definable 1-indiscernible convex sets, where $s \le 2n_p$. Using Lemmas 2.7-2.9 a number of implementation of every non-algebraic 1-type $q \in S(\emptyset)$, with $p \not\perp^w q$, $q(M')$ is also partitioned into l $\emptyset$-definable convex sets, where $l \le s$, and each of these sets is a number of implementation of one-types over $\emptyset$ in the expanded model. Define their convexity ranks:

Case 1. $U(x)$ is irrational with respect to the $E^p_{n_p-1}$-classes. Let

$$p^l_1 := \{U^l_1(x)\},\ p^r_1 := \{U^r_1(x)\}, q^l_1 := \{Q^l_1(x)\},\ q^r_1 := \{Q^r_1(x)\}$$

It is clear that $p_1^l \not\angle^w q_1^l$ and $p_1^r \not\angle^w q_1^r$ with

$$RC(p_1^l) = RC(p_1^r = n_p),\ RC(q_1^l = RC(q_1^r) = n_q$$

From this point and forward we omit the indication of weakly orthogonal pairs among these types

$$p_1^l \perp^w p_1^r,\ q_1^l \perp^w q_1^r.\ p_1^l \perp^w q_1^r,\ q_1^l \perp^w p_1^r$$

Case 2a. E_i^p-class is divided by $U(x)$ where $2 \leq i \leq n_p - 1$ and $U(x)$ is irrational with respect to the E_{i-1}^p-classes.

Suppose

$p_m^r := \{U_m^r(x)\}$ and $p_m^l := \{U_m^l(x)\}$ for $n_p - i + 1 \geq m \geq 1$.

If

(1) $RC(q) = RC(p)$ then $q_m^l := \{Q_m^l(x)\}$ and $q_m^r := \{Q_m^r(x)\}$ for $n_p - i + 1 \geq m \geq 1$. Then $p_m^l \not\angle^w q_m^l, p_m^r \not\angle^w q_m^r$, and

$$RC(p_m^l) = RC(q_m^l) = RC(p_m^r) = RC(q_m^r) = m + i - 1$$

where $1 \leq m \leq n_p - i + 1$

(2) If $RC(p) > RC(q)$. Let $j := n_p - n_q$. If $i \leq j$, then

$$q_m^l := \{Q_m^l(x)\},\ q_m^r := \{Q_m^r(x)\},\ p_m^l \not\angle^w q_m^l,\ p_m^r \not\angle^w q_m^r$$

for $j - i + 2 \leq m \leq n_p - i + 1$.

If $\exists \alpha\, from\, p(M')$ such that $dcl(\{\alpha\}) \cap q(M') \neq \emptyset$ then the formula $Q_{j-i+1}^c(x)$ additionally appears and so is the type $q_{j-i+1}^c := \{Q_{j-i+1(x)}^c\}$. It is clear that $RC(q_{j-i+1}^c) = 0$. If $dcl(\{\alpha\}) \cap q(M') = \emptyset$ for every $\alpha \in p(M')$ then this formula doesn't appear. Also

$$RC(p_m^l) = RC(p_m^r) = m + i - 1$$
for $1 \leq m \leq mn_p - i + 1$,

and

$$RC(q_m^l) = RC(q_m^r) = m + i - 1 - j$$
for $j - i + 2 \leq m \leq n_p - i + 1$.

In case of $i > j$ the situation is identical to case (1).

(3) If $RC(p) < RC(q)$. Let $j := n_q - n_p$. In this case

$$q_m^l := \{Q_m^l(x)\}, \qquad q_m^r := \{Q_m^r(x)\}, \qquad p_m^l \not\perp^w q_m^l, \qquad p_m^r \not\perp^w q_m^r$$

for $1 \le m \le n_p - i + 1$. Note that

$RC(p_m^l) = RC(p_m^r) = m + i - 1,\ m + i - 1 + j = RC(q_m^r) = RC(q_m^l)$

where $n_p - i + 1 \ge m \ge 1$.

Case 2b. $U(x)$ divides E_1^p-class. Suppose $p_m^r := \{U_m^r(x)\}$ and $p_m^l := \{U_m^l(x)\}$ where $0 \le m \le n_p - 1$.

(1) $RC(p) = RC(q)$. In this case

$$q_m^l := \{Q_m^l(x)\},\ q_m^r := \{Q_m^r(x)\},\ p_m^l \not\perp^w q_m^l,\ p_m^r \not\perp^w q_m^r$$

and

$$RC(p_m^l) = RC(q_m^l) = RC(p_m^r) = RC(q_m^r) = m + 1$$

(2) If $RC(p) > RC(q)$, then

$$q_m^l := \{Q_m^l(x)\},\ q_m^r := \{Q_m^r(x)\},\ p_m^l \not\perp^w q_m^l,\ p_m^r \not\perp^w q_m^r$$

for $j \le m \le n_p - 1$. If exists $\alpha\, from\, p(M')$ such that $dcl(\{\alpha\}) \cap q(M') \ne \emptyset$ then the formula $Q_{j-1}^c(x)$ additionally appears and so is the type $q_{j-1}^c := \{Q_{j-1}^c(x)\}$. It is clear that $RC(q_{j-1}^c) = 0$. In case of $dcl(\{\alpha\}) \cap q(M') = \emptyset$ for each $\alpha \in p(M')$ the formula doesn't appear. Note that

$$RC(p_m^l) = RC(p_m^r) = m + 1$$

for $0 \le m \le n_p - 1$, and

$$RC(q_m^l) = RC(q_m^r) = m + 1 - j$$

for $j \le m \le n_p - 1$,

(3) In case of $RC(p) < RC(q)$ Let $j := n_q - n_p$, then

$$q_m^l := \{Q_m^l(x)\},\ q_m^r := \{Q_m^r(x)\},\ p_m^l \not\perp^w q_m^l,\ p_m^r \not\perp^w q_m^r$$

for $0 \le m \le n_p - 1$.

Note that

$$RC(p_m^l) = RC(p_m^r) = m + 1,\ RC(q_m^l) = RC(q_m^r) = m + 1 + j,$$

for $0 \le m \le n_p - 1$.

Case 3. E_i^p-class is divided by $U(x)$ for some $2 \le i \le n_p$ and $U(x)$ is quasirational to the right with respect to the E_{i-1}^p-classes.

An example for the types $q_m^r, q_m^l, p_m^r, p_m^l$ where m is from interval $n_p - i + 1 \ge m \ge 1$ is similar to the case 2a. Suppose $q_0^l := \{Q_0^l(x)\}$ and $p_0^l := \{U_0^l(x)\}$. Thus $p_0^l \not\perp q_0^l$ and

$$RC(p_0^l) = RC(q_0^l) = i - 1$$

So we shown that in any expansion by a unary convex predicate there always exists at least one non-algebraic 1-type $p' \in S_1(\emptyset)$ such that $RC(p') = n_p$, moreover this convexity rank is maxiaml.Thus we conclude that T' and theory $Th(M)$ have the same convexity ranks. Also note that the (p,q)-splitting formula (or a (q,p)-splitting formula in case $RC(p) < RC(q)$) for non-weakly orthogonal types p, q is a (p', q')-

splitting formula for new non-algebraic non-weakly orthogonal types p', q'. And finally from the Theorem 2.2 it follows that T' is countably categorical.

■

Further, we use the obtained result to study the properties that are preserved during the expansions of models of a theory which is quite o-minimal and countably categorical by a predicate convex unary. Next is determined: properties that are preserved during such expansions are quite o-minimality, countable categoricity, and convexity rank.

Lemma 2.10 *Let T be quite o-minimal countably categorical theory, $M \vDash T$, non-algebraic one-types $p, q \in S_1(\emptyset)$, such that $p \not\perp^w q$. Suppose that M' is an expansion of a model M by a unary predicate $U(x)$ such that $U(M) \subset p(M)$ and $U(M)^- = p(M)^-$. Then $p(M)$ is partitioned into s convex $\emptyset$-definable sets in M' $\Leftrightarrow$ $q(M)$ is partitioned into s convex $\emptyset$-definable sets in M'.*

Proof of Lemma 2.10. As $p \not\perp^w q$ then by Theorem 2.2(iii) $RC(p) = RC(q)$ and there exists $\emptyset$-definable function $f: p(M) \to q(M)$, which is a locally monotone bijection. Let $P(x)$ be an $\emptyset$-definable formula, isolating type p. Suppose that $P(x)$ is divided into s convex $\emptyset$-definable formulas $U_1(x), \dots, U_s(x)$, selecting in $p(M)$ sets which are 1-indiscernible over $\emptyset$. Consider the following formulas:

$$S_i(x) := \exists y[U_i(y) \wedge f(y) = x], \quad 1 \leq i \leq s$$

It is evident that for each distinct pair i, j with $1 \leq i, j \leq s$, as $U_i(M) \cap U_j(M) = \emptyset$, we have $S_i(M) \cap S_j(M) = \emptyset$. Due to indiscernibility $U_i(M)$ over $\emptyset$ each $S_i(M)$ will also be 1-indiscernible..

Let $p_i := \{U_i(x)\}, q_i := \{S_i(x)\}$ for any $1 \leq i \leq s$. Then it is clear, that $p_i \not\perp^w q_i$, $RC(p_i) = RC(q_i)$ and $f: p_i(M') \to q_i(M')$ is $\emptyset$-definable bijection.

■

Thus taking in consideration Theorem 2.3 we establish the following:

Corollary 2.1 *Suppose M is a model of theory which is a quite o-minimal and countably categorical, and M' be an expansion of a model M by an arbitrary finite family of predicates convex and unary. Then M' is a model of theory which is a quite o-minimal and countably categorical of the same convexity rank.*

3 EXPANSIONS OF MODELS BY EQUIVALENCE RELATIONS

The section will examine expansions of countably categorical weakly o-minimal theories by special case of binary expansions – expansions by equivalence relation.

N is called an *1-indiscernible* structure if for every $c, d\, from\, N$ $type\ of\ c$ over empty set equals$type\ of\ d$ over empty set.

Example 3.1 [63, P. 65] Suppose $N_m := \langle \mathbb{Q}^m; <, E_1^2, E_2^2, \dots, E_{m-1}^2 \rangle$, where $\mathbb{Q}^m$ is the set of m-tuples $z = (\text{z}, \dots, z_{m-1})$ of rational numbers, ordered lexicographically by $<$, and suppose for each $k = 1, \dots, m-1$ the equivalence relation E_k be given by $E_k(z, t)$ iff $z_n = t_n$ for any $n < m - k$. Then for every k the equivalence classes of E_k are convex subsets of $\mathbb{Q}^m$. Moreover, E_{m-1} refines E_k for every $2 \le k \le m - 1$.

It can be shown N_m is an 1-indiscernible countably categorical weakly o-minimal structure and *Theory of* N_m has *RC*(N_m)=m.

Proposition 3.1 [49, P. 354] *Suppose N is an 1-indiscernible countably categorical weakly o-minimal structure of finite convexity rank. Then there is* $m\, from\, \omega$ *that N is isomorphic to* $N_m := \langle \mathbb{Q}^m; <, E_1^2, E_2^2, \dots, E_{m-1}^2 \rangle$ *(Example 6.1).*

In this case we examine only the problem of preserving both weak o-minimality and countable categoricity for expansions of models of 1-indiscernible countably categorical weakly o-minimal theories of finite convexity rank by a relation of equivalence spliting the universe of the model into infinite number of infinite convex classes.

Example 3.2 Suppose $N := \langle \mathbb{Q}, < \rangle$ is a linearly ordered structure on the rational numbers set $\mathbb{Q}$. It is evident N is a countably categorical o-minimal structure. We expand the model N by a new binary relation $E(z, t)$ next way:
suppose $N' := \langle \mathbb{Q}, <, E^2 \rangle$ is such that for every $c, from$ $\mathbb{Q}$ and exists $m\, from\, \mathbb{Z}$:

$$E(c, d) \text{ iff } (2m - 1)\sqrt{2} < c, d < (2m + 1)\sqrt{2}$$

Then it is easy to understand $E(z, t)$ is a relation of equivalence that splits $\mathbb{Q}$ into infinite number of infinite convex classes, and the E-classes are ordered by the type $\omega^* + \omega$.

It is a routine to show using simple quantifier elimination that N' is a weakly o-minimal structure. It should note that $Theory\, of\, N'$ is not countably categorical because the ordered set of integers is interpretable as N'/E.

Example 3.3 Suppose $N := \langle \mathbb{Q} \times \mathbb{Q}, <, E^2 \rangle$ is a linearly ordered structure on the set $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. The relation $E(z, t)$ is defined next way:

$$for\ \ every\ \ c = (i_1, m_1), d = (i_2, m_2) \in \mathbb{Q} \times \mathbb{Q} \quad E(c, d)\ iff\ i_1 = i_2.$$

It is evident $E(z,t)$ is a relation of equivalence that splits $\mathbb{Q} \times \mathbb{Q}$ into infinite number of infinite convex classes, and the E-classes are ordered by the type $\mathbb{Q}$.

We extend the universe $\mathbb{Q} \times \mathbb{Q}$ of the structure N by adding two elements to every E-class, which are the left and the right endpoints of the E-class. As a result, we get a new structure $N' := \langle N', <, E^2 \rangle$. Consider the reduct of the structure N' to the structure $N'' := \langle N', < \rangle$. It is evident N'' is a countably categorical o-minimal structure. Its expansion $N' := \langle N', <, E^2 \rangle$ is a countably categorical linearly ordered structure.

We consider the next formula:

$$\psi(z) := \exists t_1 \exists t_2 [t_1 < z < t_2 \ \wedge \ \forall x \forall y (t_1 \leq x < z \wedge z < y \leq t_2 \rightarrow \neg E(x,y))]$$

The formula means that z is some E-class endpoint. It should note that $\psi(N')$ is an union of infinite number of convex sets. Thus, $Theory\, M'$ is not weakly o-minimal.

Proposition 3.2 *Suppose N is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, N' is an expansion of the model N by an equivalence relation $E(z,t)$ spliting N into infinite number of infinite convex classes. For $Theory\ N'$ to be a countably categorical weakly o-minimal theory necessary and sufficient condisions are the next statements to hold:*

(1) It exists only finite number of E-classes having at least one endpoint;

(2) It exists only finite number of E-classes having an immediate predecessor or an immediate successor in the induced ordering on N/E.

Proof of the Proposition 3.2. (Necessary condition). We consider the formula $\psi(z)$ from Example 3.3. It is clear that $\psi(N)$ is finite. If $\psi(N)$ was infinite then it would contain an infinite interval because of weak o-minimality, but endpoints of infinite convex E-classes can't form an infinite interval. Therefore, $\psi(M)$ is finite, which means that it exists only finite number of E-classes having at least one endpoint.

Now we prove that condition (2) holds. Assume the contrary: it exists infinite number of E-classes having an immediate predecessor or an immediate successor.

Case 1. $\forall i \ (i < \omega) \ \exists m_1 (i \leq m_1 < \omega)$ it exists a discretely ordered chain of E-classes of length m_1.

Then it exists a model M' of the $Theory\ N'$ by compactness, in which there is an infinite discretely ordered chain of E-classes. We don't lose generality if suppose that such chain is ordered by the type ω. We consider the next formulas:

$$F_1(z,t) := E(z,t)$$

$$F_2(z,t) := F_1(z,t) \vee \forall y (z \leq y \leq t \wedge \neg F_1(z,y) \rightarrow E(y,t))$$

$$\dots \dots \dots$$

$$F_m(z,t) := F_{m-1}(z,t) \vee \forall y (z \leq y \leq t \wedge \neg F_{m-1}(z,y) \rightarrow E(y,t)), \quad m < \omega$$

It should note that $F_1(c, M')$ defines the class $E(c, M')$, and $F_m(c, M')$ for every $m < \omega$ defines the class $E(c, M')$ and the $m - 1$ E-classes immediately following it.

Then we obtain that there exists $c\ from\ M'$ such that

$$F_1(c, M') \subset F_2(c, M') \subset \cdots \subset F_m(c, M') \subset \cdots$$

which contradicts countable categoricity of $Theory\ N'$.

Case 2. $\exists i$ $(i < \omega)$ with a discretely ordered chain of E-classes of length i, and i is maximal with this property.

At that point $\exists\ j$ (j$< \omega$) such that $2 \leq j \leq n$ and there is an infinitely many chains of length j. Thereat $\exists c\ from\ N'$ such that $F_j(c, N')$ is a union of infinite number of convex sets, which controverts weak o-minimality of $Theory\ N'$.

(Sufficient condition) According to (1) it exists only finite number of E-classes which have at least one endpoint. Now, each the endpoint is definable in obedience to the linear ordering of N'. According to (2) we can also define using a formula every E-class which has an immediate successor or an immediate predecessor, as well as possible nonempty intervals between some of these classes (those intervals in which E-classes are densely ordered without endpoints); In addition, minimal E-classes (the leftmost E-class) or maximal E-classes (the rightmost E-class) in intervals with dense ordering of E-classes are distinguished. In consequence, we obtain finitely many $\emptyset$-definable formulas $h_k(z), 1 \leq n \leq m$, so that for all $1 \leq k < n \leq m$

$$h_k(N') \cap h_n(N') = \emptyset.$$

All the formulas define some 1-type over $\emptyset$. Using standard methods it is not difficult to understand that up to atomic formulas and the formulas $h_1(z), h_2(z), \ldots, h_m(z)$ Theory N'admits quantifier elimination (the last formulas define convex sets in N'), therefore we get that $Theory\ N'$ is a countably categorical weakly o-minimal theory. ■

Corollary 3.1 *Suppose N is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, N' is an expansion of the model N using an relation of equivalence $E(x, y)$ which split N into infinite number of infinite convex classes. Now for N' to be an 1-indiscernible countably categorical weakly o-minimal structure necessary and sufficient condisions are :*

(1) All E-classes have no endpoints in N';

(2) The induced order on E-classes is a dense linear order without endpoints.

Example 3.4 Suppose $N' := \langle \mathbb{Q}, <, E^2 \rangle$ is the structure from Example 3.2. We replace all points $c\ from\ \mathbb{Q}$ using copy of rational numbers and define a new structure $N'' := \langle \mathbb{Q} \times \mathbb{Q}, <, E^2, E_0^2 \rangle$, in which the relation $E_0(z, t)$ defines by next way:

$$E_0(c,d)\ if\ only\ if\ i_1 = i_2\quad for\ \ all\quad c = (i_1, m_1), b = (i_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$$

Now it is easy to show that $E_0(z,t)$ is a relation of equivalence which splits each E-class into infinite number of infinite convex classes that the E_0-subclasses of each E-class are densely ordered without endpoints.

It can be proved that N'' is a weakly o-minimal structure, but the $Theory\ (N'')$ is not countably categorical.

Example 3.5 Suppose $N' := \langle \mathbb{Q}, <, E^2 \rangle$ is ordered by type $\mathbb{Q}$ countable number of copies of the structure from Example 3.2. Now we get a new structure $N'' := \langle \mathbb{Q} \times \mathbb{Q}, < E^2, E_1^2 \rangle$, in which the relation $E_1(x,y)$ is defined by next way:

$$E_1(c,d)\ if\ only\ if\ i_1 = i_2\quad for\ \ any\quad c = (i_1, m_1), d = (i_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$$

At that point $E(c, N'') \subset E_1(c, N'')$ for all $c\ from\ N''$ and $E_1(z,t)$ is a relation of equivalence which splits N'' into infinite number of infinite convex classes, ordered by type $\mathbb{Q}$. Observe that E-subclasses of each E_1-class are ordered by type $\omega^* + \omega$.

It is also easy to understand that N'' is a weakly o-minimal structure, but Theory N'' is not countably categorical.

Theorem 3.1 *Suppose N is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank* m, *and* $E_1(z,t), E_2(z,t), \ldots, E_{m-1}(z,t)$ *are* $\emptyset$-*definable relations of equivalence which split N into infinite number of infinite convex classes*, *that for all* $c\ from\ N$

$$E_1(c,N) \subset E_2(c,N) \subset \cdots \subset E_{m-1}(c,N).$$

Suppose N' is a model N expansion using a new relation of equivalence $E^*(z,t)$ splitting N' into infinite number of infinite convex classes. Now for $Theory\ N'$ to be a countably categorical weakly o-minimal theory *necessary and sufficient condisions are :*

(A) Only finite number of E^*-classes having at least one endpoint exist;

(B) Only finite number of E^*-classes having an immediate predecessor or an immediate successor in the induced ordering on M/E^* exist;

(C) N' is splitted into finite number of infinite convex sets $X_1, \ldots, X_i$ such that for each $1 \le k \le i$ exactly one of the next items holds:

$(1)_s$ $\exists s\ 1 \le s \le m-1$ that $E^*(c,N) = E_s(c,N)$ for all $c\ from\ X_k$;

$(2)_1$ $E^*(c,N) \subset E_1(c,N)$, $sup\,E^*(c,N) < sup\,E_1(c,N)$ and

$$\inf E_1(c,N) < \inf E^*(c,N)\ \text{ for all }\ c\ from\ X_k;$$

$(2)_{n+1}$ $\exists n$ $1 \le n \le m-2$ that $E_n(c,N) \subset E^*(c,N) \subset E_{n+1}(c,N)$,

$$sup\, E_n(c,N) < sup\, E^*(c,N), \inf\ E_n(c,N) < \inf\ E^*(c,N),$$

$$sup\, E^*(c,N) < sup\, E_{n+1}(c,N) \quad and \quad \inf\ E_{n+1}(c,N) < \inf\ E^*(c,N)$$
for all $c\ from\ X_k$;

$(2)_m$ $E_{m-1}(c,N) \subset E^*(c,N)$, $sup\, E_{m-1}(c,N) < sup\, E^*(c,N)$ and
$\inf\ E^*(c,N) < \inf E_{m-1}(c,N)\ \ for\ all\ \ c\ from\ X_k$;

$(3)_s$ $\exists s$ $1 \le s \le m-1$ that $X_k = E^*(c,N)$ for some $c\ from\ N$ $E^*(a,M) \subset E_s(c,N)$ and

$$either \quad sup\, E^*(c,N) = sup\, E_s(c,N) \quad or \quad \inf\ E^*(c,N) = \inf\ E_s(c,N);$$

$(4)_s$ $\exists s$ $1 \le s \le m-1$that $X_k = E_s(c,N)$ for some $c\ from\ N$, $E_s(c,N) \subset E^*(c,N)$ and

$$either \quad sup\, E_s(c,N) = sup\, E^*(c,N) \quad or \quad \inf\ E_s(c,N) = \inf E^*(c,N);$$

$(5)_s$ $\exists s$ $1 \le s \le m-1$ such that $X_k = E^*(c,N) \cap E_s(c,N)$ for some $c\ from\ N$,

$E^*(c,N) \backslash E_s(c,N) \neq \emptyset$ and $E_s(c,N) \backslash E^*(c,N) \neq \emptyset$.

Proof of Theorem 3.1. ($Necessary\ condision$) (A) and (B) ensue from the Proposition 3.2 proof. Further, we consider the following formulas for any $1 \le s \le m-1$ and $1 \le n \le m-2$:

$$\psi_{1s}^{\emptyset}(z) := \forall t[E_s(z,t) \leftrightarrow E^*(z,t)]$$

i.e. $\psi_{1s}^{\emptyset}(z)$ defines the set of elements $c\ from\ N$ that $E^*(c,N) = E_s(c,N)$.

$$\psi_{21}^{\emptyset}(z) := \forall t[E^*(z,t) \to E_1(z,t)] \wedge \exists x_1 \exists x_2 (x_1 < z < x_2 \wedge E_1(x_1,x_2) \wedge$$

$$\wedge \neg E^*(z_1,z) \wedge \neg E^*(z,x_2))]$$

i.e. $\psi_{21}^{\emptyset}(z)$ defines the set of elements $c\ from\ N$ such that $E^*(c,N) \subset E_1(c,N)$, $sup\, E^*(c,N) < sup\, E_1(c,N)$ and $\inf E_1(c,N) < \inf\ E^*(c,N)$.

$$\psi_{2,n+1}^{\emptyset}(z) := \forall t[(E_n(z,t) \to E^*(z,t)) \wedge (E^*(z,t) \to E_{n+1}(z,t))] \wedge$$

$$\wedge \exists y_1 \exists x_1 \exists x_2 \exists y_2 (y_1 < x_1 < z < x_2 < y_2 \wedge E_{n+1}(y_1,y_2) \wedge E^*(x_1,x_2) \wedge$$

$$\wedge \neg E^*(y_1, x_1) \wedge \neg E^*(x_2, y_2) \wedge \neg E_n(x_1, z) \wedge \neg E_j(z, x_2))]$$

i.e. $\psi^{\emptyset}_{2,n+1}(x)$ defines the set of elements $c\ from\ N$ such that $E_n(c, N) \subset E^*(c, N) \subset E_{n+1}(c, N)$, $sup\, E_n(c, N) < sup\, E^*(c, N)$, $\inf E_n(c, N) < \inf E^*(c, N)$, $sup\, E^*(c, N) < sup\, E_{n+1}(c, N)$ and $\inf\ E_{n+1}(c, N) < \inf E^*(c, N)$.

$$\psi^{\emptyset}_{2n}(z) := \forall t[E_{m-1}(z, t) \rightarrow E^*(z, t)] \wedge$$

$$\wedge \exists x_1 \exists x_2 (x_1 < z < x_2 \wedge E^*(x_1, x_2) \wedge \neg E_{m-1}(x_1, z) \wedge \neg E_{m-1}(z, x_2))]$$

i.e. $\psi^{\emptyset}_{2m}(z)$ defines the set of elements $c\ from\ N$ such that $E_{m-1}(c, N) \subset E^*(c, N)$, $sup\, E_{m-1}(c, N) < sup\, E^*(c, N)$ and $\inf E^*(c, N) < \inf E_{m-1}(c, N)$.

$$\psi^{r}_{3s}(z) := \forall t[E^*(z, t) \rightarrow E_s(z, t)] \wedge \forall t_1[z < t_1 \wedge \neg E^*(z, \mathrm{t}) \rightarrow \neg E_s(z, t_1)]$$

i.e. $\psi^{r}_{3s}(z)$ defines classes $E^*(c, N)$ for some element $c\ from\ N$ such that

$$E^*(c, N) \subset E_s(c, N) \quad and \quad sup\, E^*(c, N) = sup\, E_s(c, N).$$

$$\psi^{l}_{3s}(z) := \forall t[E^*(z, t) \rightarrow E_s(z, t)] \wedge \forall t_1[z > t_1 \wedge \neg E^*(z, t_1) \rightarrow \neg E_s(z, t_1)]$$

i.e. $\psi^{l}_{3s}(z)$ defines classes $E^*(c, N)$ for some element $c\ from\ N$ such that

$$E^*(c, N) \subset E_s(c, N) \quad and \quad \inf E^*(c, N) = \inf E_s(c, N).$$

$$\psi^{r}_{4s}(z) := \forall t[E_s(z, t) \rightarrow E^*(z, t)] \wedge \forall t_1[z < t_1 \wedge \neg E_s(z, t_1) \rightarrow \neg E^*(z, t_1)]$$

i.e. $\psi^{r}_{4s}(z)$ defines an $E_s(c, N)$-classes for an element $c\ from\ N$ that

$$E_s(c, N) \subset E^*(c, N) \quad and \quad sup\, E_s(c, N) = sup\, E^*(c, N).$$

$$\psi^{l}_{4s}(z) := \forall t[E_s(z, t) \rightarrow E^*(z, t)] \wedge \forall t_1[z > t_1 \wedge \neg E_s(z, t_1) \rightarrow \neg E^*(z, t_1)]$$

i.e. $\psi^{l}_{4s}(z)$ defines an $E_s(c, N)$-classes for an element $c\ from\ N$ that

$$E_s(c, N) \subset E^*(c, N) \quad and \quad \inf E_s(c, N) = \inf E^*(c, N).$$

$$\psi^{r}_{5s}(z) := \exists t_1 \exists t_2 [t_1 < z < t_2 \wedge E_s(t_1, z) \wedge E^*(z, t_2) \wedge \neg E_s(z, t_2) \wedge \neg E^*(t_1, z)]$$

i.e. $\psi^{r}_{5s}(z)$ defines non-empty intersections $E^*(c, N) \cap E_s(c, N)$ for some $c\ from\ N$ that there exist $d_1 \in E_s(c, N) \backslash E^*(c, N)$ and $d_2 \in E^*(c, N) \backslash E_s(c, N)$ where $d_1 < d_2$.

$\psi^{l}_{5s}(z) := \exists t_1 \exists t_2 [t_1 < z < t_2 \wedge E^*(t_1, z) \wedge E_s(z, t_2) \wedge \neg E^*(z, t_2) \wedge \neg E_s(t_1, z)]$

i.e. $\psi^{l}_{5s}(x)$ defines non-empty intersections $E^*(c,N) \cap E_s(c,N)$ for some $c\ from\ N$ such that there exist $d_1\ from\ E_s(c,N) \backslash E^*(c,N)$ and $d_2\ from\ E^*(c,N) \backslash E_s(c,N)$ with $d_1 > d_2$.

We can understand that for all $c\ from\ N'$ there exist k, s, ε such that $1 \leq k \leq 5$, $1 \leq s \leq m-1$, ε from $\{\emptyset, r, l\}$ and $N' \vDash \psi^{\varepsilon}_{is}(c)$, and also that

$$N' \vDash \neg \exists z [\psi^{\varepsilon_1}_{ks}(z) \wedge \psi^{\varepsilon_2}_{nj}(z)]$$

for all single $k, n, s, j, \varepsilon_1, \varepsilon_2$ that $1 \leq k, n \leq 5$, $1 \leq s, j \leq m-1$, $\varepsilon_1, \varepsilon_2\ from\ \{\emptyset, r, l\}$ provided that $k \neq n$, $s \neq j$ or $\varepsilon_1 \neq \varepsilon_2$.

In accodance with weak o-minimality of $Theory\ N'$, each of these formulas defines a set that is the union of finitely many convex sets, which implies (C).

($Sufficient\ condision$) In obedience to the proof of Proposition 6.2 performance of (A) and (B) conditions define using a formula the available endpoints of E-classes; any E-class, which has an immediate successor or an immediate predecessor, as well as gaps where E-classes are densely ordered without endpoints; in addition, minimal or maximal E-classes are distinguished in the intervals of dense ordering of E-classes having the leftmost or rightmost E-class. Therefore, we obtain finitely many $\emptyset$-definable formulas $h_k(z)$,$1 \leq n \leq m$, that for each $1 \leq k < n \leq m$

$$h_k(N') \cap h_n(N') = \emptyset.$$

The performance of condition (C) provides that any formulas $\psi^{\varepsilon}_{ks}(z)$, $1 \leq k \leq 6$, $1 \leq s \leq m-1$, $\varepsilon\ from\ \{\emptyset, r, l\}$, defines a set that is the union of finitely many convex sets. Then by linear ordering of N' all formulas $\psi^{\varepsilon}_{is}(z)$ decomposes into finitely many convex $\emptyset$-definable formulas $\psi^{\varepsilon 1}_{ks}(z), \psi^{\varepsilon 2}_{ks}(z), \ldots, \psi^{\varepsilon m^{\varepsilon}_{ks}}_{ks}(z)$ for some $m^{\varepsilon}_{ks} < \omega$.

It is clear that there exist only finite nuber of E^*-classes (and, therefore, $\exists s$ $1 \leq s \leq m-1\ :\ E_s$-classes) defined by next formulas:

$$\psi^{r}_{6s}(z) := \exists x (z < x \wedge \neg E_s(z, x) \wedge \forall y_1 (E^*(x, y_1) \rightarrow \neg E_s(z, y_1)) \wedge$$

$$\wedge \forall y_2 (E_s(z, y_2) \rightarrow \neg E^*(x, y_2)) \wedge \forall v [z \leq v \leq x \rightarrow E_s(z, v) \vee E^*(v, x)])$$

i.e. $\psi^{r}_{6s}(z)$ defines classes $E_s(c,N)$ for some $c\ from\ N$ which are immediately followed by an E^*-class.

$$\psi^{l}_{6s}(z) := \exists x (z < x \wedge \neg E^*(z, x) \wedge \forall y_1 (E_s(x, y_1) \rightarrow \neg E^*(z, y_1)) \wedge$$

$$\wedge \forall y_2 (E^*(z, y_2) \rightarrow \neg E_s(x, y_2)) \wedge \forall v [z \leq v \leq x \rightarrow E^*(z, v) \vee E_s(v, x)])$$

i.e. $\psi^{r}_{6s}(z)$ defines classes $E^{*}(c, N)$ for some $c\, from\, N$ which are immediately followed by an E_s-class.

Suppose opposite: there exist infinitely many E^{*}-classes and E_s-classes defined by the formula $\psi^{l}_{6s}(z)$. Then we sate that $\psi^{l}_{6s}(N')$ is the union of infinitely many disjoint convex sets. Actually, if $d\, from\, \psi^{l}_{6s}(\mathrm{N}')$, then there exists $c\, from\, N'$ such that

$$E^{*}(c, N') \cap E_s(c, N) = \emptyset$$

and $d < c$. As far as the condition (B) holds, we have that it exists an infinite convex part of $E_s(c, N')$ satisfying the formula $\psi^{\emptyset}_{2s}(z)$, whence $\psi^{\emptyset}_{2s}(N')$ is the union of infinitely many disjoint convex sets, but it contradicts the condition (C).

At the end, it easy to proved using standard methods that Theory N' accepts quantifier elimination up to atomic formulas and formulas

$$h_1(z), h_2(z), \dots, h_m(z), \psi^{\varepsilon 1}_{11}(z), \dots, \psi^{\varepsilon m^{\varepsilon_1}_{1,m-1}}_{1,m-1}(z); \dots; \psi^{\varepsilon 1}_{51}(z), \dots, \psi^{\varepsilon m^{\varepsilon}_{5,m-1}}_{5,m-1}(z),$$

from which we have Theory N' is a countably categorical weakly o-minimal theory. ■

Corollary 3.2 *Suppose N is an 1-indiscernible countably categorical weakly o-minimal structure of convexity rank n, and $E_1(z,t), E_2(z,t), \dots, E_{m-1}(z,t)$ are $\emptyset$-definable equivalence relations splitting N into an infinite number of infinite convex classes, such that for all c from N*

$$E_1(c, N) \subset E_2(c, N) \subset \cdots \subset E_{m-1}(c, N).$$

Suppose N' is model M expansion by a new relation of equivalence $E^{*}(z,t)$ which splits N' into infinite number of infinite convex classes. Thereat for N' to be an 1-indiscernible countably categorical weakly o-minimal structure *necessary and sufficient condisions are*:

(A) every E^{*}-class has not endpoints in N';

(B) the induced order on E^{*}-classes is a dense linear order without endpoints;

(C) for all *c from* N' exactly one of the following $2m - 1$ items holds:

$(1)_s$ $\exists s$ $(1 \le s \le m-1)$ that $E^{*}(c, N) = E_s(c, N)$;

$(2)_1$ $E^{*}(c, N) \subset E_1(c, N)$, $sup\, E^{*}(c, N) < sup\, E_1(c, N)$ and
$\inf E_1(c, N) < \inf E^{*}(c, N)$;

$(2)_{n+1}$ $\exists n$ $1 \le n \le m-2$ that $E_n(c, N) \subset E^{*}(c, N) \subset E_{n+1}(c, N)$,
$sup\, E_n(c, N) < sup\, E^{*}(c, N), \inf E_n(c, N) < \inf E^{*}(c, N)$,

$$sup\, E^*(c,N) < sup\, E_{n+1}(c,N) \quad and \quad \inf E_{n+1}(c,N) < \inf \; E^*(c,N);$$

$(2)_m$ $E_{m-1}(c,N) \subset E^*(c,N)$, $sup\, E_{m-1}(c,N) < sup\, E^*(c,N)$ and

$$\inf E^*(c,N) < \inf E_{m-1}(c,N).$$

4 EXPANSIONS OF MODELS BY ARBITRARY BINARY PREDICATES

The section is devoted to investigation of the question of properties preservation when expanding models of countably categorical weakly ordered-minimal theories by arbitrary binary predicates. Earlier we have studied the problem of preserving properties for expansions of models of countably categorical weakly o-minimal theories by unary predicates. As it is known, in work [8, P. 1382] B.S. Baizhanov proved that the expansion of a model of a weakly o-minimal theory by a unary predicate that distinguishes a finite number of convex sets preserves weak o-minimality of the expanded theory. However, in the case of expanding a model of a weakly o-minimal theory by a binary predicate that distinguishes a finite number of convex sets for each fixed both the first and the second parameter, the expanded theory can lose weak o-minimality (Example 4.1).)

Example 4.1 Suppose $N := \langle \mathbb{R}, < \rangle$ is a linearly ordered structure on the set of real numbers $\mathbb{R}$. It is evident that N is a model of a countably categorical o-minimal theory. We expand the model N by a new binary relation $S(z,t)$ next way: suppose $N' := \langle \mathbb{R}, <, S^2 \rangle$ be such that $S(z,t)$ is the graph of the next unary function f, defined as $f(d) = 2d$ for every d *from* $\mathbb{Q}$ and $f(a) = -a$ for every $a\, from\, \mathbb{R} \backslash \mathbb{Q}$. It is evident that $S(c,N)$ and $S(N,c)$ for every $c\, from\, N$ are singleton sets, i.e. convex sets. However, notice that N' is not weakly o-minimal, so far as there is no partition of the set $\mathbb{R}$ into a finitely many convex sets, on each of which the definable function f is locally constant or locally monotone.

Example 4.2 Suppose $N := \langle \mathbb{Q}, < \rangle$ is a linearly ordered structure on the set of rational numbers $\mathbb{Q}$. It is evident that N is countably categorical 1-indiscernible o-minimal structure. Consider a binary predicate $R(z,t)$ expansion of a structure N: denote as $N' := \langle \mathbb{Q}, <, R^2 \rangle$ that for all $c, d\, from\, \mathbb{Q}$

$$R(c,d) \Leftrightarrow c \leq d < c + \sqrt{2}.$$

We understand that $R(c,N')$ and $R(N',c)$ are convex for all $c\, from\, N'$. It easy to prove that N' is weakly o-minimal 1-indiscernible structure.

The formula $F(z,\mathrm{t}) := R(t,z)$ is convex to the right p-stable, where

$$p(z) := \{z = z\} \in S_1(\emptyset).$$

It is obvious that the formula $F(z,t)$ doesn't generate equivalence.

Study next formulas:

$$R_2(z,t) := \exists y[R(z,y) \wedge R(y,t)],$$

$$R_m(z,t) := \exists y[R_{m-1}(z,y) \wedge R(y,t)], m \geq 2$$

For every $c\ from\ N'$ we got

$$R(c,N') \subset R_2(c,N') \subset \cdots \subset R_n(c,N') \subset \cdots,$$

It means that Theoty N′ is not countably categorical.

Suppose N is a weakly o-minimal structure, $C \subseteq N$, a non-algebraic type $p \in S_1(C)$, an C-definable formula $R(z,t)$, that is p-preserving, that is for all c *from* $p(M)$ there exist elements $d_1, d_2\ from\ p(M)$ that the next holds $d_1 < R(N,c) < d_2$.

Owing to the weak o-minimality of M the set $R(N,c)$ composes of a union of convex sets, whose quantity is finite. It is evident that every single of sets mentioned is $C \cup \{c\}$-definable. To the left of element a there is a finite quantity of such definable convex sets. Signify them as $R_1^l(z,t), \dots, R_s^l(z,t)$, in the following way

$$R_s^l(N,c) > R_{s-1}^l(N,c) > \cdots > R_1^l(N,c) \geq c.$$

In the same way to the right of element a there is a finite quantity of such definable convex sets. Signify them as $R_1^r(z,t)$, ..., $R_i^r(z,t)$, in the following way

$$c \leq R_1^r(N,c) < R_2^r(N,c) < \cdots < R_i^r(N,c).$$

In the event that definable convex set has an element c. Denote the set as $R^a(z,t)$. Thereby if $R^a(N,c) \neq \emptyset$, then $\exists\ d_1, d_2 from\ R^a(N,c)$ so that $d_1 < c < d_2$.

Determine next formulas:

$$F^a(z,t) := t \leq z \wedge R^a(z,t)$$

$$G^a(z,t) := t \geq z \wedge R^a(z,t)$$

$$F_k^r(z,t) := t \leq z \wedge \forall y[R_k^r(y,t) \rightarrow z < y], 1 \leq k \leq i$$

$$F_k^{r*}(z,t) := t \leq z \wedge \exists y[R_k^r(y,t) \wedge z \leq y], 1 \leq k \leq i$$

$$G_n^l(z,t) := t \geq z \wedge \forall y[R_n^l(y,t) \rightarrow y < z], 1 \leq k \leq s$$

$$G_n^{l*}(z,t) := t \geq z \wedge \exists y[R_n^l(y,t) \wedge y \leq z], 1 \leq k \leq s$$

It is evident that the formulas $F^a(z,t), F_k^r(z,t), F_k^{r*}(z,t), 1 \leq k \leq i$, are convex to the right p-stable and formulas $G^a(z,t), G_n^l(z,t), G_n^{l*}(z,t), 1 \leq n \leq s$ are convex to the left p-stable.

A formula $R(z,t)$ is said to be *equivalence generated*, if all nontrivial formulas from the set

$\Delta := \{F^a(z,t),\ F_k^r(z,t),\ F_k^{r*}(z,t),\ G^a(z,t),\ G_n^l(z,t),\ G_n^{l*}(z,t) | 1 \leq k \leq i, 1 \leq$

$n \leq s\}$

are equivalence generating formulas.

Example 4.3 Suppose $N := \langle \mathbb{Q} \times \mathbb{Q}, < \rangle$ is a linearly ordered structure on the set $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically. It is evident that N is countably categorical o-minimal structure.

Represent the next binary formulas $E(z,t)$ and $R_1(z,t)$ on the set $\mathbb{Q} \times \mathbb{Q}$: for all $c = (i_1, m_1), d = (i_2, m_2) \in \mathbb{Q} \times \mathbb{Q}$

$$E(c,d) \ iff \ i_1 = i_2$$

$$R_1(c,d) \ \ iff \ i_1 = i_2 \wedge m_1 \leq m_2 < m_1 + \sqrt{2}$$

Suppose

$$R(z,t) := t \leq z \wedge E(z,t) \wedge \neg R_1(z,t)$$

and $N' := \langle \mathbb{Q} \times \mathbb{Q}, <, R^2 \rangle$ is an expansion of model M using binary predicate $R(z,t)$. It is evident that for all $c \ from \ N' \ R(N',c)$ is convex and $c < R(N',c)$.

It easy to understand that N' is 1-indiscernible weakly o-minimal structure, nevertheless Theory N$'$ is not countably categorical theory.

Examine the next formulas:

$$F_1(z,t) := t \leq z \wedge \forall y[R(y,t) \to z < y]$$

$$F_2(z,t) := t \leq z \wedge \exists y[R(y,t) \wedge z \leq y]$$

The formulas $F_1(z,t), F_2(z,t)$ are p-stable convex to the right, where

$$p(z) := \{z = z\} \in S_1(\emptyset),$$

$F_2(z,t)$ is equivalence-generating, and $F_1(z,t)$ is not equivalence-generating. Therefore the predicate $R(z,t)$ is not equivalence-generating predicate.

Theorem 4.1 *Suppose N is 1-indiscernible countably categorical weakly o-minimal structure of convexity rank 1, N' is 1-indiscernible weakly o-minimal expansion of a structure N using binary predicate $R(z,t)$.*

Thereat for $Theory \, N'$ to be countably categorical *necessary and sufficient condisions is* thefulfillment next properties:

(1) $R(z,t)$ and $L(z,t) := R(t,z)$ is equivalence generated;

(2) For each empty definable equivalence relation $E(z,t)$, generated by a predicate $R(z,t)$, the set of E-classes is ordered densely.

Proof of Theorem 4.1. *(Necessary condision).* Let $Theory \, N'$ be countably categorical theory. Examine a predicate $R(z,t)$. Through weak o-minimality of the structure N' for every $c\ from\ N'$ $R(N',c)$ and $R(c,N')$ are a union finite quantity of convex sets. According to Proposition 2.2 both formulas $R(z,t)$ and $L(z,t)$ must be equivalence-generated.

Suppose $E(z,t)$ is a random empty definable equivalence relation. Through 1-indiscernibility the set of E-classes must be either densely ordered without endpoints, or discretely ordered without endpoints. Hence, by countable categoricity, the set of E-classes must be densely ordered.

(S*ufficient condisions).* Suppose $R(z,t)$ and $L(z,t)$ are formulas generated equivalence. Examine $E^*(z,t)$ a random $\emptyset$-definable equivalence relation, generated by a predicate $R(z,t)$. In accordance with condition, the set of E^*-classes is densely ordered. Through 1-indiscernibility there is neither the leftmost E^*-class, neither rightmost E^*-class. Also through 1-indiscernibility there doesn't exist E^*-class, having at least one endpoint (if every E^*-class would have at least one endpoint, then we would get contradiction to weak o-minimality of N').

Through the weak o-minimality of a structure N' for all elements $c\ from\ N'$ the sets $R(N',c)$ and $R(c,N')$ are unions of finite quantity of convex sets. Hence, it exists only a finite quantity of formulas of the form $F^a(z,t)$, $F_k^r(z,t)$, $F_k^{r*}(z,t)$, $G^a(z,t)$, $G_n^l(z,t)$, $G_n^{l*}(z,t)$, $1 \leq k \leq m_1$, $1 \leq n \leq m_2$ for some $m_1, m_2 < \omega$. As in accordance with the condition $R(z,t)$, $L(z,t)$ are equivalence generated formulas, then every non-trivial formula from next:

$$\Delta := \{F^a(z,t), F_k^r(z,t), F_k^{r*}(z,t), G^a(z,t), G_n^l(z,t),$$

$$G_n^{l*}(z,t) | 1 \leq k \leq m_1, 1 \leq n \leq m_2\}$$

generates an equivalence relation. Thereby we get only finite quantity of $\emptyset$-definable equivalence relations, generated by predicate $R(z,t)$.

Suppose $\{E_1(z,t), E_2(z,t), \dots, E_n(z,t)\}$ are a complete set of $\emptyset$-definable equivalence relations, generated by predicate $R(z,t)$. Through 1-indiscernibility there don't exist k,n such that $k \neq n$, $1 \leq k,n \leq m$ and for some $c\ from\ N'$ $E_k(a,M') \subset E_n(c,N')$,

$$sup\, E_k(c,N') = sup\, E_n(c,N') \quad or \quad \inf E_k(c,N') = \inf E_n(c,N').$$

Also there don't exist $k,n\ from\ \{1,\dots,m\}$ such that for some $c\ from\ N'$

$$E_k(c,N') \backslash E_n(c,N') \neq \emptyset \quad and \quad E_n(c,N') \backslash E_k(c,N') \neq \emptyset.$$

Onward for some $1 \leq k,n \leq m$ if there is $c\ from\ N'$ such that $E_k(c,N') \subseteq E_n(c,N')$, then for all $c\ from\ N'$ $E_k(c,N') \subseteq E_n(c,N')$. Thereby it exists $1 \leq i \leq m$ (possible situation when for some $k,n\ from\ \{1,\dots m\}$ $E_k(c,N') = E_n(c,N')$) and perhaps some renumbering of the existing equivalence relations so that for all $c\ from\ N'$ we would have

$$E_1(c,N') \subset E_2(c,N') \subset \cdots \subset E_i(c,N').$$

As far as, in accordance with condition, the set of E-classes is densely ordered for each $\emptyset$-definable equivalence relation $E(z,t)$, then E_k-subclasses of every E_{k+1}-class are densely ordered without endpoints, where $0 \leq k \leq i$ and

$$E_0(z,t) := z = t, \quad E_{i+1}(z,t) := z = z \wedge t = t.$$

Onward we can establish using standard methods, that $Theory\ N'$ accepts quantifier elimination up to atomic formulas and formulas $E_k(z,t)$, $1 \leq k \leq i$, where do we get that $Theory\ N'$ is countably categorical.

Example 4.4 shows, that there exists an expansion by binary predicate, that preserves weakly o-minimality but doesn't preserve countable categoricity.

Example 4.4 Consider a structure $M := \langle M, <, P^1, f^1 \rangle$ that is linearly ordered, the universe is defined as following $M = P(M) \cup \neg P(M), P(M) < \neg P(M)$, where $P(M)$ and $\neg P(M)$ are matched with $\mathbb{Q}$ and have the same order as $\mathbb{Q}$, where $\mathbb{Q}$ is a set of rational numbers. To distinguish the elements of these sets, for any element $a \in P(M)$ denote its identical element in $\neg P(M)$ as a'.

An unary function with D$om\ (f) = P(M)$ and R$ange\ (f) = \neg P(M)$ is defined by the symbol f as following $f(a) = a'$, so strictly increasing bijection is defined from $P(M)$ to $\neg P(M)$ by f. It can be shown that M is a weakly o-minimal ω-categorical structure.

Consider model M expansion by new binary relation $R(x,y)$. Consider $M' := \langle M, <, P^1, f^1, R^2 \rangle$, where for any $a \in P(M'), b \in \neg P(M')$

$$R(a,b) \Leftrightarrow b < a' + \sqrt{2}$$

It clear, that M' is still weakly o-minimal structure. Examine next formulas:

$$F_1(x,y) := y \leq x \wedge \exists t(f(t) = y \wedge R(t,x)) \wedge \neg P(x)$$

$$F_n(x,y) := \exists t(F_{n-1}(t,y) \wedge F_1(x,t)), n \geq 2$$

and for any $b \in P(M)$

$$F_1(M'b) \subset F_2(M',b) \subset \ldots \subset F_n(M',b) \subset \ldots,$$

so *Theory M'* isn't countably categorical theory.

Example 4.5 Consider a linearly ordered structure $M := \langle M, <, P^1, R_1^2 \rangle$ such that the universe $M = P(M) \cup \neg P(M), P(M) < \neg P(M)$, with $P(M)$ and $\neg P(M)$ matched with $\mathbb{Q}$ and had the same order as $\mathbb{Q}$, where $\mathbb{Q}$ is a set of rational numbers.

To distinguish the elements of these sets, for any element $a \in P(M)$ settle it's identical element in $\neg P(M)$ as a'. Symbol R_1 is designate by next way: for any $a \in P(M), b \in \neg P(M)$

$$R_1(a,b) \Leftrightarrow b < a' + \sqrt{2}$$

It can be shown, that M is weakly o-minimal ω-categorical structure.

Also examine the model M expansion M' using $R_2(x,y)$ binary relation: suppose $M' := \langle M, <, P^1, R_1^2, R_2^2 \rangle$, in which for all $a \in P(M'), b \in \neg P(M')$

$$R_2(a,b) \Leftrightarrow b < a' + 2\sqrt{2}$$

It is clear, that M' remains weakly o-minimal structure. Examine next formula:

$$F_1(x,b) := b \leq x \wedge \neg P(x) \wedge \exists t(P(t) \wedge \neg R_1(t,b) \wedge R_2(t,b) \wedge R_2(t,x))$$

Defining $F_n(x,y)$ formulas analogically to previous example we obtain that $Th(M')$ is non countably categorical.

The following example shows that under the expansion of a weakly o-minimal ω-categorical structure by (p,q)-splitting formula for non-algebraic types $p, q \in S_1(\emptyset)$ we can lose as 1-indiscernibility of this types and weakly o-minimality of such expansion.

Example 4.6 [47, P. 1511] Consider a linearly ordered structure $M := \langle M, <, P^1, f^1 \rangle$, where M is a union of interpretations P and $\neg P$ that are disjoint and $P(M) < \neg P(M)$, with $P(M)$ identified with the $\mathbb{Q} \times \mathbb{Q}$, ordered lexicographically, and $\neg P$ identified with $\mathbb{Q}$ and had the same order as $\mathbb{Q}$, where $\mathbb{Q}$ is a set of rational numbers. We define using simbol f a partial unary function for which $Dom(f) = P(M)$ and $Range(f) = \neg P(M)$ by next way: $f((n,m)) = n$, for every $(n,m)\ from\ \mathbb{Q} \times \mathbb{Q}$.

It is obvious that Theory M is a weakly o-minimal ω-categorical theory, and

$$E(z,t) := P(z) \wedge P(t) \wedge \exists y(\neg P(y) \wedge f(z) = y \wedge f(t) = y)$$

determine a relation of equivalence, splitting $P(M)$ into infinitely many convex classes.

Let $p(x) := \{P(x)\}, q(x) := \{\neg P(x)\}$. Then it's clear that $p, q \in S_1(\emptyset)$ are non-algebraic, $RC(p) = 2, RC(q) = 1, p \not\perp^w q$.

Consider the following formulas:

$$\Phi_1(x,y) := \neg P(x) \wedge P(y) \wedge \forall t(E(t,y) \rightarrow f(t) < x)$$

$$\Phi_2(x,y) := \neg P(x) \wedge P(y) \wedge \exists t(t \geq y \wedge f(t) \leq x)$$

It is clear that formulas $\Phi_1(x,y)$ and $\Phi_2(x,y)$ are (q,p)-splitting formulas and $\Phi_1(b,M) \subset \Phi_2(b,M)$ for any $b \in q(M)$.

Consider an expansion M' of a structure M by binary predicate $R(x,y)$ such that for every $(a,c) \in P(M)$ and $b \in \neg P(M)$ the formula below is true:

$$R(b,(a,c)) \Leftrightarrow f((a,c)) \leq b \wedge (f((a,c)) = b \Rightarrow c < b + \sqrt{2})$$

Then the set $R(b,M)$ is convex for every $b \in q(M)$,

$$R(b,M) \subset q(M), \quad R(b,M)^{-} = p(M)^{-}$$

and $\Phi_1(b,M) \subset R(b,M) \subset \Phi_2(b,M)$.

Consider the following formula:

$$\theta(x) := P(x) \wedge \exists (f(x) = t \wedge R(t,x))$$

It is evident, that the set $\theta(M')$ is the union of infinite number of $\neg\theta(M')$-separable convex sets, so $p(M')$ is not 1-indiscernible and M' and non weakly o-minimal.

There was given a complete description of countably categorical theories which have finite convexity rank in the work [56, P. 606]. Since in weakly o-minimal theory of 1 rank of convexity there is no equivalence relation with infinite number of convex classes, then as a corollary we get:

Corollary 4.1 *Consider a weakly o-minimal countably categorical theory T of convexity rank 1, $M \vDash T$, $|M| = \omega$. The following is true:*

(i) Exists an infinite set $C = \{c_0, \ldots, c_n\} \subseteq M$ ($M \cup \{-\infty, +\infty\}$, if M is not having the last or the first elements), made up of every single empty definable element from M (except $-\infty, +\infty$ if there is some of them) that $M \vDash c_i < c_j$ for all $i < j < n$ and for each $j \in \{1, \ldots, n\}$ either $M \vDash \neg\exists x c_{j-1} < x < c_j$ or $I_j\{x \in M : M \vDash c_{j-1} < x < c_j\}$ is linear order, that is dense, without endpoints and there exists $k_j \in \omega$ and $p_1^j, \ldots, p_{k_j}^j \in S_1(\emptyset)$ such that $I_j = \bigcup_{s=1}^{k_j} p_s^j(M)$;

(ii) For any nonalgebraic types $p, q \in S_1(\emptyset)$ such that $p \not\perp^w q$

If for some $\alpha \in p(M)$ $dcl(\{\alpha\}) \cap q(M) \neq \emptyset$, then there is the unique empty definable function $f: p(M) \to q(M)$, such that it is strictly monotone bijection on $p(M)$

If $dcl(\{\alpha\}) \cap q(M) = \emptyset$ for all $\alpha \in p(M)$, then there exists the only (p,q)-splitting formula $S(x,y)$, such that $f(x) := sup \; S(x,M)$ - strictly monotone on $p(M)$

Thus a theory T accepts exception of quantifiers to the language

$$\{=,<\} \cup \{c_i : i \le n\} \cup \left\{U_s(x) : s \le r = \sum_{j=1}^{n} k_j\right\} \cup$$

$$\cup \{f_{i,j} : dcl(\{\alpha\}) \cap p_j(M) \neq \emptyset \ \ for\ some \ \ \alpha \in p_i(M)\} \cup$$

$$\cup \{S_{i,j}(x,y) : p_i \not\angle^{w} p_j, dcl(\{\alpha\}) \cap p_j(M) = \emptyset$$

$$for\ all \ \ \alpha \in p_i(M), S_{i,j}(x,y) \ \ --- \ \ (p_i, p_j) - splitting\ formula\,\}$$

Where $U_s(x)$ isolates type p_s for every $s \le r$

Moreover for any ordering with selected elements, as represented in (i)-(iii), related to weakly o-minimal ω-categorical theory of convexity rank 1, as it was presented earlier.

Suppose M is weakly o-minimal, countably categorical of convexity rank 1, p and q are nonalgebraic 1-types in M over emptyset. Suppose M' is an expansion of a structure M by binary predicate $R(x,y)$, such that for any $\alpha \in p(M)$ the set $R(\alpha, M)$ is convex, $R(\alpha, M) \subset q(M)$ and $R(\alpha, M)^- = q(M)^-$. Suppose also that M' is weakly o-minimal structure and $p, q \in S_1^{M'}(\emptyset)$.

The following theorem gives necessary and sufficient conditions of theory $Th(M')$ countable categoricity.

Theorem 4.2 Consider a model M *of a weakly o-minimal ω-categorical theory of 1 rank of convexity, non-algebraic one-types* p *and* q *are from* $S_1^M(\emptyset)$. *Let* M' *be an weakly o-minimal expansion of convexity rank 1 of a structure* M *by binary predicate* $R(x,y)$, *such that* $p, q \in S_1^{M'}(\emptyset)$, *for any* $\alpha \in p(M')$ *the set* $R(\alpha, M')$ *is convex,* $R(\alpha, M') \subset q(M')$ *and* $R(\alpha, M')^- = q(M')^-$. *ThereatTheory* M'*is countably categorical* $\Leftrightarrow p \perp_M^w q$.

Proof: (⇒) Suppose $Theory\ M'$ is ω-categorical. By countable categoricity $Th(M')$ there exists an empty definable formulas $U_p(x)$ and $U_q(x)$, such that $U_p(M') = p(M')$ and $U_q(M') = q(M')$.

Note that as $Th(M)$ is weakly o-minimal, countably categorical theory of convexity rank 1, by corollary 7.1 for any $p, q \in S_1^M(\emptyset)$ either these types are weakly orthogonal or there exists the only $\emptyset$-definable bijection and there is no other relations between them or there exits the only one (p,q)-splitting formula and there is no other relations.

Towards a contradiction suppose that types p, q are nonweakly orthogonal in structure M. Hence there exists (p,q)-splitting formula $R_1(x,y)$, so for any $\alpha \in p(M')$ the set $R_1(a, M')$ is convex, $R_1(a, M) \subset q(M')$ and $R_1(a, M')^- = q(M')^-$.

As theory $Th(M)$ has convexity rank 1, then the function $f_1(x) := sup \ \ R_1(x, M)$ is strictly monotone on $p(M)$. Without loosing the generality suppose that f_1 is strictly increasing on $p(M)$.

Let $f_2(x) := sup \ \ R(x, M')$. As $Th(M')$ has convexity rank 1 also, $p, q \in S_1^{M'}(\emptyset)$, then f_2 is also strictly monotone on $p(M')$.

Case 1. f_2 is strictly increasing on $p(M')$.

Without loosing the generality suppose, that for some $\alpha \in p(M')$ the following holds $f_1(\alpha) < f_2(\alpha)$. Then for any $\alpha \in p(M')$ also $f_1(\alpha) < f_2(\alpha)$ holds.

Consider the following formulas:

$$F_!(x,y) := y \leq x \land \neg U_q(x) \land \exists t(U_p(t) \land \neg R_1(t,y) \land R(t,y) \land R(t,x))$$

$$F_n(x,y) := \exists t(F_{n-1}(t,y) \land F_1(x,t)), n \geq 2$$

Then for any $b \in \neg U_q(M')$ the following holds:

$$F_1(M',b) \subset F_2(M',b) \subset \ldots \subset F_n(M',b) \subset \ldots$$

so $Theory\ M'$ is not countably categorical. It contradicts suggestion.

Case 2. Suppose f_2 strictly decreases on $p(M')$.

Thereat $\exists \alpha, \alpha' from\, p(M')$ that $f_1(\alpha) < f_2(\alpha)$ and $f_1(\alpha') > f_2(\alpha')$. The last contradicts that $p \in S_1^{M'}(\emptyset)$

($\Leftarrow$) Let $p \perp_M^w q$. Let's show $Theory\ M'$ is an ω-categorical.

Case 1. For some $\alpha \in p(M')$ the set $R(\alpha, M')$ has right endpoint.

So consider the formula below:

$$\theta(x) := U_q(x) \land \exists u(U_p(u) \land \forall y(R(u,y) \to y \leq x) \land \forall t(\neg R(u,t) \to t \geq x))$$

It is clear that $\theta(M') \subseteq q(M')$. By the assumptions $\theta(M') \neq \emptyset$, from where we get that $\theta(M') = q(M')$. Thus for any $\alpha \in p(M')$ the set $R(a, M')$ has the right endpoint.

Hence the function $f_2(x) := sup\ \ R(x, M')$ is strictly monotone bijection between $p(M')$ and $q(M')$.

Case 2. For some $\alpha \in p(M')$ the set $R(\alpha, M')$ does not have right endpoint.

So consider the formula below:

$$\psi(x) := U_p(x) \land \forall y_1(R(x,y_1) \to \exists t_1(y_1 < t_1 \land R(x,t_1))) \land$$

$$\forall y_2(\neg R(x,y_2) \to \exists t_2(t_2 < y_2 \land \neg R(x,t_2)))$$

It is clear that $\psi(M') \subseteq p(M')$. By the assumptions $\psi(M') \neq \emptyset$, from where we get that $\psi(M') = p(M')$. Thus for any $\alpha \in p(M')$ the set $R(a, M')$ does not have the right endpoint.

hence the formula $R(x,y)$ is the only (p,q)-splitting formula.

In both cases $Th(M')$ admits quantifier elimination to language

$$\{=,<\} \cup \{c_i : i \leq n\} \cup \{U_s(x) : s \leq r\} \cup$$

$$\cup \{f_{i,j} : dcl(\{\alpha\}) \cap p_j(M') \neq \emptyset\ \ for\ \ some\ \ \alpha \in p_i(M')\} \cup$$

$\cup \{R_{i,j}(x,y): p_i \not\angle_{M'}^{w} p_j, dcl(\{\alpha\}) \cap p_j(M') = \emptyset \quad for \ all \quad \alpha \in p_i(M')\}$
Where $\{c_0, \ldots, c_n\}$ is set of all $\emptyset$-definable elements of structure M'. $U_s(x)$ isolates nonalgebraic 1-type p_s for every $s \leq r$; $f_{i,j}$ is the only $\emptyset$-definable bijection between $p_i(M')$ and $p_j(M')$, which is strictly monotone. $R_{i,j}(x,y)$ is the only (p_i, p_j)-splitting formula. Then by the corollary 7.1 theory $Th(M')$ is countably categorical. ■

The following example shows that an expansion of ω-categorical weakly ordered minimal structure of convexity rank greater than 1 by (p,q)-splitting formula for nonalgebraic types $p, q \in S_1(\emptyset)$ can preserve countable categoricity even in the case when this types are non weakly orthogonal.

Example 4.7 Revising example 4.6 introduce the following changes: The interpretation of P is Q^3 ordered lexicographically, and let partial unary function f is defined by equality $f((n,m,k)) = n$ for every $(n,m,k) \in Q^3$.

It is evident, that M is similarly weakly ordered minimal ω-categorical structure.

By the assumptions for each $\beta \in q(M)$ there is $a \in Q$ such that $f\big((a, Q \times Q)\big) = \beta$ where $\{a\} \times Q \times Q \subseteq P(M)$.

Consider an expansion M' of structure M by binary predicate $R(x,y)$ such that for any $(a,c,d) \in P(M)$ and $\beta \in \neg P(M)$ the following holds:

$R(\beta, (a,c,d)) \Leftrightarrow f(a, Q \times Q)) \leq \beta \wedge (f((a, Q \times Q)) = \beta \Rightarrow (a,c,d) \in (a,c,Q))$

Then we also have that the set $R(\beta, M)$ is convex for every $\beta \in q(M)$

$$R(\beta, M) \subset q(M), \quad R(\beta, M)^- = p(M)^-$$

and $\Phi_1(\beta, M) \subset R(\beta, M) \subset \Phi_2(\beta, M)$.

It is possible to see $Theory\ M'$ is weakly ordered minimal ω-categorical theory and $p, q \in S_1^{M'}(\emptyset)$.

Examine the formula below:

$$F(x,y) := y \leq x \wedge P(y) \wedge P(x) \wedge \exists t(\neg P(t) \wedge R(t,x))$$

It is possible to see, that $F(x,y)$ is convex to the right p-stable formula, moreover formula $F(x,y)$ is equivalence-generating. Thus by Lemma 2.4 the formula

$$E'(x,y) := F(x,y) \vee F(y,x)$$

is a relation of equivalence which splits $p(M')$ into infinitely many infinite classes and for all type realisation $\alpha\, from\, p(M')$ the extending is faithful $E'(\alpha, M') \subset E(\alpha, M')$. Wherefrom we have that convexity rank of type p equals to 3 and this implies that the theory $Th(M')$ has RC=3.

5 EXTERNAL DEFINABILITY AND MODEL COMPLETENESS

Let $\mathfrak{M} = \{\langle M, \Sigma \rangle\}$, $\mathfrak{N} = \{\langle N, \Sigma \rangle\}$ be two structures of the signature Σ, such that $\mathfrak{M} \prec \mathfrak{N}$. We say that a set $A \subset M$ is *externally definable*, if $A = \phi(N, \bar{\alpha}) \cap M$ for some N-formula, $\bar{\alpha} \in N \backslash M$. In stable theory any externally definable set is internally definable. If we take some family of externally definable sets such that it closed over all set theory operations: union, intersection, taking complement, cartesian product, projection, we obtain family of externally definable sets. Expansion by such family of externally definable sets is called to be *pure externally definable expansion*. Notice that main moment in the expansion by externally definable sets to be pure externally definable expansion, is that this family closed over the operation of projection [39, P. 5435] (preprint 1994).

The idea to consider the expansion by externally definable set belongs to Dugald Macpherson, David Marker and Charles Steinhorn and expansion of model by externally definable set was introduced for the first time in 1994 in preprint of [39, P. 5435]

5.1 Model completeness

Definition 5.1.1 *Suppose $\mathfrak{M} := \langle M; \Sigma \rangle$ is a model of weakly ordered minimal theory T, $\mathfrak{N}$ are its large satiated elementary extension of $\mathfrak{M} \prec \mathfrak{N}$, $A \subseteq M$.*

We say that an one-type $p \in S_1(A)$ is solitary if any A-2-formula $\varphi(x, y)$, such that for any $\alpha \in p(N)$, if $\varphi(\alpha, N) \subset p(N)$ then $\varphi(\alpha, N) = \{\alpha\}$. In case when $\mathfrak{M}$ is a model of o-minimal theory, $A = M$, p defines irrational cut, this one-type is called to be uniquely realizable [40, P. 63].

We say that an one-type $p \in S_1(A)$ is quasi-solitary if there is a 2-formula $\theta(x, y)$ over A that for every $\alpha \in p(N)$, $\theta(\alpha, N) \subset p(N)$ and for every 2-formula $\varphi(x, y)$ over M, that for every $\alpha \in p(N)$, whenever $\varphi(\alpha, N) \subset p(N)$ the $\varphi(\alpha, N) \subseteq \theta(\alpha, N)$. In the event when $\theta(x, y) \equiv x = y$ we have solitary one-type.

Notice that it follows from the definition that $\theta(N, \alpha)$ is convex.

The following theorem was proved for an o-minimal case in [39, P. 5435].

Theorem 5.1.1 [39, P. 5435] *Let $\mathfrak{M}$ be a model of a weakly ordered minimal theory such that any type over M is solitary. Then any expansion of $\mathfrak{M}$ by a unary convex predicate is pure externally definable expansion and has weakly o-minimal theory.*

Proof: Consider $\mathfrak{M} := \langle M; \Sigma \rangle$ - a model of a weakly ordered minimal theory of signature Σ, $\Sigma^+ := \Sigma \cup \{P^1\}$ be the expanded signature of $\mathfrak{M}^+ := \langle M; \Sigma^+ \rangle$, and T^+ be the theory of $\mathfrak{M}^+$.

Convex expansion always go through in general two cuts. Expansion of model by convex predicate can be considered step by step as expansion through each cut. We don't lose generality if we consider such expansion, that next sentence holds:

$$\mathfrak{M}^+ \vDash \forall x \forall y (P(x) \wedge (y < x \rightarrow P(y))).$$

Let $\mathfrak{N}$ be an elementary extension of $\mathfrak{M} \prec \mathfrak{N}$ such that there exists a realization α of one-type over M which defines cut of expansion. So, we suppose that $P(\mathfrak{M}^+) = (-\infty, \alpha)_{\mathfrak{N}} \cap M$.

Claim: For any $\psi^+(t_1, t_2, \ldots, t_n)$ of signature Σ^+ $\exists$ $K_{\psi^+}(t_1, t_2, \ldots, t_n, \alpha)$ formula of the signature Σ so that for every $a_1, a_2, \ldots, a_n \in M$ the proceeding is true:

$$[\mathfrak{M}^+ \vDash \psi^+(a_1, a_2, \ldots, a_n) \leftrightarrow \mathfrak{N} \vDash K_{\psi^+}(a_1, a_2, \ldots, a_n, \alpha)]$$

Proof of Claim: It will be examined the claim by induction method on depth of formulas constructions.

Consider the depth 0 formula. For it

$$\psi_0^+(t_1, t_2, \ldots, t_n) = \vee [\wedge \psi_i(t_1, t_2, \ldots, t_n) \wedge P(t_j) \wedge \neg P(t_k)],$$

where ψ_i are the signature Σ formulas. For the ψ_0^+ formula we can define

$$K_{\psi_0^+}(t_1, t_2, \ldots, t_n, \alpha) := \vee [\wedge \psi_i(t_1, t_2, \ldots, t_n) \wedge (t_j < \alpha) \wedge (t_k > \alpha)]. \quad (1)$$

Then for any $a_1, a_2, \ldots, a_n \in M$ the following holds:

$$[\mathfrak{M}^+ \vDash \psi_0^+(a_1, a_2, \ldots, a_n) \leftrightarrow \mathfrak{N} \vDash K_{\psi_0^+}(a_1, a_2, \ldots, a_n, \alpha)].$$

Suppose that relation holds for the formulas of depth n. Consider the following formula

$$K_{\exists y \psi^+(y, \bar{t})}(\bar{t}, \alpha) := \exists z_1 \exists z_2 ((z_1 < \alpha < z_2) \wedge \exists y \forall z ((z_1 < z < z_2) \rightarrow K_{\psi^+}(y, \bar{t}, z))). \quad (2)$$

Further proof is identical to a given in historical review for o-minimal theories by Macpherson-Marker-Steinhorn.

Theorem 5.1.2 [88] *(Model completeness of expansion by unary predicate, solitary)*

Let $\mathfrak{M} := \langle M; \Sigma \rangle$ be a model of weakly o-minimal model complete theory. Expansion of model $\mathfrak{N}$ by unary predicate $\mathfrak{M}^+ := \langle M; \Sigma^+ \rangle$, where $\Sigma^+ := \Sigma \cup \{P^1\}$, T^+ is theory of $\mathfrak{M}^+$.

Let $\mathfrak{L}^+ \vDash T^+$, such that $\mathfrak{M}^+ \subset \mathfrak{L}^+$, since T is model complete, $\mathfrak{M} \prec \mathfrak{L}$.

If $P^1(x)$ goes through solitary type then $\mathfrak{M}^+ \prec \mathfrak{L}^+$. Thus T^+ is model complete.

Proof: Let $U(x, y, \bar{z})$ be convex to the right formula for any $\bar{z}$ such that

$$\forall\bar{z}\forall x\forall y(U(x, y, \bar{z}) \to (x < y \wedge \forall u(x < u < y \to U(x, u, \bar{z})))).\ (^{*})$$

Claim: Theory
$T^{+} \vDash$ for any $\bar{z}$ and some x $(P(x) \wedge$ any $x'(x < x'$
$\to$ it exists y $(U(x', y, \bar{z}) \wedge \neg P(y)))$.

Proof of Claim: We will prove using method by contradiction. Suppose that the statement for T^{+} is not true, then
$T^{+} \vDash$ for some $\bar{z}$ and any $x(P(x) \to$ for some $x'(x <$
$x' \wedge$ forany $y(U(x', y, \bar{z}) \to P(y)))$.
As far as $T^{+} = Th(\mathfrak{M})$ for some $\bar{b} \in M$ we have

$$\mathfrak{M}^{+} \vDash \forall x(P(x) \to \exists x'(x < x' \wedge \forall y(U(x', y, \bar{b}) \to P(y))))\ (^{**}).$$

Let $C := P(\mathfrak{M}^{+}) \cap M$, $D := \neg P(\mathfrak{M}^{+}) \cap M$. Then $C < D$ and $M = C \cup D$. Consider set of M-1-formulas $\Gamma(x) := \{(c < x < d \wedge \forall y(U(x, y, \bar{b}) \to y < d)) | c \in C, d \in D\}$. By $(^{**})$, $\Gamma(x)$ is consistent and has unique extension to complete solitary one-type over M, which determines cut and the expansion. So any realization of Γ satisfies solitary one-type one side and $\{\alpha\} \neq U(\alpha, N, \bar{b}) \subset p(N)$ on other side by $(^{*})$. Contradiction.

■

Claim says that for any model $\mathfrak{L}^{+}$ of T^{+}, the cut determined on $\mathfrak{L}$ is solitary. Since $\mathfrak{M}^{+} \subset \mathfrak{L}^{+}$, $P(\mathfrak{M}^{+}) \subset P(\mathfrak{L}^{+})$. The last means that $C_M \subset C_L < D_L$, $D_M \subset D_L$. Thus we can take the realization α of the cut for $\mathfrak{L}^{+}$ as in proof of Theorem 8.1 and convenable for the proof for $\mathfrak{M}^{+}$. For any $\varphi^{+}(\bar{x})$ of theory T^{+} and $\bar{a} \in M$ that $\mathfrak{M}^{+} \vDash \varphi^{+}(\bar{a})$ there is $K_{\varphi^{+}}(\bar{x}, y)$ of theory T and $\alpha \in N \backslash M$ such that

$$\mathfrak{M}^{+} \vDash \varphi^{+}(\bar{a}) \leftrightarrow \mathfrak{N} \vDash K_{\varphi^{+}}(\bar{a}, \alpha).$$

The same holds for $\mathfrak{L}^{+}$:

$$\mathfrak{L}^{+} \vDash \varphi^{+}(\bar{b}) \leftrightarrow \mathfrak{N} \vDash K_{\varphi^{+}}(\bar{b}, \alpha).$$

$K_{\varphi^{+}}(\bar{x}, \alpha)$ is the same formula for both $\mathfrak{M}$ and $\mathfrak{L}$. Whenever we consider $\bar{a} \in M$

$$\mathfrak{M}^{+} \vDash \varphi^{+}(\bar{a}) \Leftrightarrow \mathfrak{N} \vDash K_{\varphi^{+}}(\bar{a}, \alpha) \Leftrightarrow \mathfrak{L}^{+} \vDash \varphi^{+}(\bar{a}),$$

thus T^{+} is model complete.

■

Theorem 5.1.3 [89] *(Model completeness of expansion by unary predicate, quasisolitary) Let $\mathfrak{M} := \langle M; \Sigma \rangle$ be a model of a weakly o-minimal, model complete* *theory,* $\mathfrak{M}^+ := < M; \Sigma^+ >$ *be an expansion of model* $\mathfrak{M}$ *by unary predicate* P^1*, such that* $P^1(x)$ *goes through quasi-solitary type, where* $\Sigma^+ := \Sigma \cup \{P^1\}$*. Then*

(i) $T^+ = Th(\mathfrak{M}^+)$ is weakly o-minimal and this expansion of $\mathfrak{M}$ is pure externally definable expansion.

(ii) T^+ is model complete, if greatest p-preserving 2-formula $\emptyset$-definable

(iii) $T^+ \cup r^+(\bar{c}) = T(\bar{c})^+$ model complete theory, where $\bar{c} \in M$ is tuple from the greatest p-preserving 2-formula.

Proof: (i) This repeats proof of the same theorem for case solitary. The proof consists in showing by induction on the construction of a formula in the expanded language $\varphi^+(x_1, x_2, \dots, x_n)$ that the hold on the parameters is equivalent to the hold of the corresponding formula

$K_{\varphi^+}(x_1, x_2, \dots, x_n, \alpha, \bar{c})$ in the initial language on these parameters and an additional parameter from the saturated elementary extension. For any $\varphi^+(x_1, x_2, \dots, x_n)$ of signature Σ^+ it exists $K_{\varphi^+}(x_1, x_2, \dots, x_n, \alpha, \bar{c})$ formula with $\bar{c}\ from\ M$, $\alpha\ from\ N \backslash M$ of the signature Σ so that for all $a_1, a_2, \dots, a_n\ from\ M$ the next holds:

$$[\mathfrak{M}^+ \vDash \varphi^+(a_1, a_2, \dots, a_n) \leftrightarrow \mathfrak{N} \vDash K_{\varphi^+}(a_1, a_2, \dots, a_n, \alpha, \bar{c})].$$

There are small change in the formulas (1) and (2). Let $E(x, y, \bar{c})$ be the greatest p-preserving formula. If in solitary case $P(\mathfrak{M}^+) = (-\infty, \alpha)_N \cap M$ i.e. by formula $x < \alpha$, then in case quasi-solitary by formula $x < E(\alpha, N, \bar{c})$.

$$K_{\varphi_0^+}(x_1, x_2, \dots, x_n, \alpha, \bar{c}) := \vee\ [\wedge\ \varphi_i(x_1, x_2, \dots, x_n) \wedge (x_j < E(\alpha, N, \bar{c}) \wedge (x_k > E(\alpha, N, \bar{c})]. \qquad (1')$$

$$K_{\exists y \varphi^+(y, \bar{x})}(\bar{x}, \alpha, \bar{c}) := \exists z_1 \exists z_2 ((z_1 < E(\alpha, N, \bar{c}) < z_2) \wedge \exists y \forall z((z_1 < z < z_2) \rightarrow K_{\varphi^+}(y, \bar{x}, z))). \qquad (2')$$

(ii), (iii) Let $U(x, y, \bar{z})$ be convex to right formula of signature Σ satisfied $(^*)$, then the property "$U(x, y, \bar{a})$ is p-preserving" is expressed by Σ^+-formula:

$$H^+(U, \bar{a}) := \exists x_1 (P^1(x_1) \wedge \forall x_2 ((x_1 < x_2 \wedge P^1(x_2)) \rightarrow (U(x_2, M, \bar{a}) \subset P^1(M)))).$$

$\Theta^+(E, \bar{c}, U)$ be $\Sigma^+(\bar{c})$-formula that says that E is more than any U p-preserving:

$$H^+(E, \bar{c}) \wedge \forall \bar{z} (H^+(U, \bar{z}) \rightarrow \exists x_1 (P^1(x_1) \wedge \forall x_2 ((x_1 < x_2 \wedge P^1(x_2)) \rightarrow (U(x_2, M) \subseteq E(x_2, M, \bar{c}))))).$$

Let B the set of all convex to right Σ-formulas $U(x, y, \bar{z})$ with condition $(^*)$. Then define two sets of Σ^+-sentences. $\Gamma_0 := \{\exists \bar{v} \Theta^+(E, \bar{v}, U) | U \in B\}$ and $\Gamma := \{\Theta^+(E, \bar{c}, U) | U \in B\}$. Since for any $U(x, y, \bar{z})$ convex to right Σ-formula holds:

$\mathfrak{M}^+ \vDash \Theta^+(E,\bar{c},U)$. $\Gamma_0 \subset Th(\mathfrak{M}^+) = T^+$ and $\Gamma \subset Theory(\mathfrak{M}^+(\bar{c})) = T^+ \cup r^+(\bar{c})$, whereat $\mathfrak{M}^+(\bar{c}) = \langle M, \Sigma^+, \bar{c}\rangle$, whereat the parameters $\bar{c}$ is the ones of the most great p-preserving 2-formula. So far as $\mathfrak{M}^+ \vDash H^+(E,\bar{c},E)$, if we consider arbitrary model $\mathfrak{L}^+$ of T^+ as an expansion of $\mathfrak{L}$ then we obtain that this expansion goes through quasi-solitary 1-type with greatest p-preserving $E(x,y,\bar{b})$, $\bar{b} \in L$. Then for any $\varphi^+(x_1,x_2,\dots,x_n)$ of the Σ^+signature $\exists$ $K_{\varphi^+}(x_1,x_2,\dots,x_n,\alpha,\bar{b})$ formula with $\bar{c}\ from\ M$, $\alpha\ from\ N\backslash M$ of the Σ signature so that for all $a_1,a_2,\dots,a_n \in L$ the proceeding is true:

$$[\mathfrak{L}^+ \vDash \varphi^+(a_1,a_2,\dots,a_n) \leftrightarrow \mathfrak{N} \vDash K_{\varphi^+}(a_1,a_2,\dots,a_n,\alpha,\bar{b})].$$

(ii) Consider $\mathfrak{L}^+ := \langle L;\Sigma^+\rangle \vDash T^+$ such that $\mathfrak{M}^+ \subset \mathfrak{L}^+$. Since T is model complete $\mathfrak{M} \prec \mathfrak{L} \prec \mathfrak{N}$. If $E(x,y)$ is $\emptyset$-definable Σ-formula, then for any $\varphi^+(x_1,x_2,\dots,x_n)$ of signature Σ^+, for any $a_1,a_2,\dots,a_n \in M$, because the greatest p-preserving in both models coincide

$$\mathfrak{M}^+ \vDash \varphi^+(a_1,a_2,\dots,a_n) \leftrightarrow \mathfrak{N} \vDash K_{\varphi^+}(a_1,a_2,\dots,a_n,\alpha) \leftrightarrow \mathfrak{L}^+ \vDash \varphi^+(a_1,a_2,\dots,a_n).$$

Thus T^+ is model complete.

(iii) Consider $\mathfrak{L}^+(\bar{c}) := \langle L;\Sigma^+,\bar{c}\rangle \vDash T^+(\bar{c})$ such that $\mathfrak{M}^+(\bar{c}) \subset \mathfrak{L}^+(\bar{c})$. Since T is model complete $\mathfrak{M} \prec \mathfrak{L} \prec \mathfrak{N}$, $\mathfrak{M}(\bar{c}) \prec \mathfrak{L}(\bar{c})$. If $E(x,y,\bar{c})$ is $\bar{c}$-definable Σ-formula, then for any $\varphi^+(x_1,x_2,\dots,x_n)$ of signature Σ^+, for any $a_1,a_2,\dots,a_n \in M$, because the greatest p-preserving in both models coincide

$$\mathfrak{M}^+ \vDash \varphi^+(a_1,a_2,\dots,a_n) \leftrightarrow \mathfrak{N} \vDash K_{\varphi^+}(a_1,a_2,\dots,a_n,\alpha,\bar{c}) \leftrightarrow \mathfrak{L}^+ \vDash \varphi^+(a_1,a_2,\dots,a_n).$$

Thus $T^+(\bar{c})$ is model complete. ■

5.2 External definability

Externally definable sets is an special case of expansion, which is equivalent to extension, i.e. any formula in new language is defined in initial language using new external parameters.

External definability. Let $\mathfrak{M}$ be elementary substructure of $\mathfrak{N}$. The pair of models $\mathfrak{M},\mathfrak{N}$, such that $\mathfrak{N}$ is saturated over M, is called *beautiful* pair. Let $p := tp(\alpha|M)$, where $\bar{\alpha} \in N\backslash M$. Then we define the predicate $R_{(\psi,p)}(\bar{y})$ on the set M, where $\psi(\bar{x},\bar{y})$ is arbitrary formula, $\vDash R_{(\psi,p)}(\bar{a}) \Leftrightarrow$ if the next holds:

$$\psi(\bar{x},\bar{a}) \in tp(\bar{\alpha} \in M) \quad if \ and \ only \ if \quad \mathfrak{N} \vDash \psi(\bar{\alpha},\bar{a})$$

Let $\Sigma^+ := \{R_{(\psi,p)}(\bar{y}) | p \in S(M),\ \psi \in \Sigma\}$ denote by $\mathfrak{M}^+ = \langle M;\Sigma^+\rangle$. If a pair of models (M, N) is conservative pair (that is type of any tuple elements from N over M is definable), then from the definition the structure $\mathfrak{M}^+$ is the structure obtained from $\mathfrak{M}$ scolemization of $\mathfrak{M}$. We will consider two approaches of the simple cases when $\mathfrak{M}^+$ constructed from one 1-type for o-minimal theory.

Consider the Σ signature complete theory T. Denote a model of the theory as $\mathfrak{M}$. The $\mathfrak{M}$ expansion using type$p \in S_1(M)$ is $\mathfrak{M}_p^+$, if $\mathfrak{M}_p^+ := \langle M; \Sigma_p^+ \rangle$, where $\Sigma_p^+ := \{R_{(\psi,p)}(\bar{y}) | \psi \in \Sigma\}$.

If for every single formula $\phi(\bar{y})$ of the signature Σ_p^+ there exists a formula $K_\phi(\bar{y}, \bar{z})$ of the signature Σ and $\exists\ \bar{\alpha}\, from\, N \backslash M$ such that $\forall\ \bar{a}\, from\, M$ the proceeding is true:

$$\mathfrak{M}_p^+ \vDash \phi(\bar{a}) \Leftarrow \vDash K_\phi(\bar{a}, \bar{\alpha}).$$

We say it this occasion that $\mathfrak{M}_p^+$ *admits uniformly representation of* Σ_p^+-formulas by Σ-formulas,

In 2000 [39, P. 5435] it was verified that unary convex predicate expansion of o-minimal structure preserves weak o-minimality, in the case when this predicate is traversed by a 1-type, that has a unique realisation, by Macpherson-Marker-Steinhorn. Uniquely realizable 1-type $p \in S_1(M)$ over M, as introduced by D. Marker [40, P. 63], is the only the 1-type p realization from the set of the type realization over a prime model M and this realisation of type. The following characteristic is true for uniquely realizable 1-type p: for the number of p realizations there is not any definable functions that can act on it. At the same time Macpherson-Marker-Steinhorn approach was descirbed in historical review, and was applied for uniquelly realisable, solitary, quasi-solitary, model complete theories.

B.S. Baizhanov approach. For the event when $p \in S_1(M)$ is a non uniquely realizable type, on the base of theory of (non)orthogonality of 1-types and its systematization made in [25, P. 565; 40, P. 63; 41, P. 146; 42, P. 185] (Marker, Mayer, Pillay-Steinhorn, Marker-Steinhorn, 1986–1994), B.S. Baizhanov proposed [44; P. 3] (1995) to take the constants from an infinite indiscernible sequence $I = \langle \alpha_n \rangle_{n<\omega}$ over M for $K_{\exists x \psi(x,\bar{y})}$, where α_n from $p(\mathfrak{N})$. Taking attention that there is a finite quantity of irrational cuts (that is 1-types over M) ifthe set $K_{\psi(x,\bar{y})}(\mathfrak{N}, \bar{a}, \bar{\alpha}_n) \cap M = \emptyset$, that for every such 1-type $r \in S_1(M)$, $K_{\psi(x,\bar{y})}(\mathfrak{N}, \bar{a}, \bar{\alpha}_n)$ is a subset of

$QV_r(\bar{\alpha}_n) := \{\beta \in r(\mathfrak{N})$ | there exists an $M\bar{\alpha}_n$-1-formula $\Theta(x, \bar{\alpha}_n)$, such that $\beta \in \Theta(\mathfrak{N}, \bar{\alpha}_n) \subset r(\mathfrak{N})\}$.

There is two parts of the idea to use an indiscernible sequence.

(i) If $\exists\ \ c\, from\, M, \mathfrak{N} \vDash K_{\psi(x,\bar{y})}(c, \bar{a}, \bar{\alpha}_n)$, consequently for every tuple $\bar{\gamma}$ equals to $\langle \alpha_{i_0}, ..., \alpha_{i_n} \rangle$ $(n < i_0 < \cdots < i_n)$, $\mathfrak{N} \vDash K_{\psi(x,\bar{y})}(c, \bar{a}, \bar{\gamma})$, when $\bar{\alpha}_n$ and $\bar{\gamma}$ have identic type over M.

(ii) But to find a sequence I such that for each $r \in S_1(M)$, for every $\bar{\gamma} = \langle \alpha_{i_0}, \dots, \alpha_{i_n} \rangle$ $(n < i_0 < \dots < i_n)$, $QV_r(\bar{\alpha}_n) \cap QV_r(\bar{\gamma}) = \emptyset$.

To find the sequence I that is indiscernible define the properties (P_1)-(P_5) that follow from the theory of non orthogonality of 1-types over sets in o-minimal theories and the systematization of 1-types.

(P_1) [40, P. 63] (Marker 1986). Consider two one types $r, q \in S_1(A)$, and let type $r(y) \cup q(x)$ be non complete 2-type (In 1978 Shelah verified that such q and r are non weakly orthogonal). In this case there exists an monotonic bijection $g: q(\mathfrak{N}) \to r(\mathfrak{N})$ definable over A and therefore, q is irrational whenever r is irrational;

Moreover q is uniquely realizable whenever r is.

Recall that whenever type $q \in S_1(A)$ is irrational $q(\mathfrak{N})$ is convex non-definable set without maximal and minimal elements.

(P_2) Whenever $q \in S_1(A)$ is irrational for any $\bar{\gamma}$, $QV_q(\bar{\gamma}) = V_q(\gamma)$, where $V_q(\bar{\gamma}) := \{\beta \in q(\mathfrak{N}) \mid \exists \delta_1, \delta_2 \in q(\mathfrak{N}),$ *there exists an* $A\bar{\gamma}$-1-formula $S(x, \bar{\gamma})$, $\delta_1 < S(\mathfrak{N}, \bar{\gamma}) < \delta_2, \beta \in S(\mathfrak{N}, \bar{\gamma})\}$.

The *quasi-neighborhood of* $\bar{\gamma}$ in q $(QV_q(\bar{\gamma}))$ is the union of sets definable over $A\bar{\gamma}$, and every this definable set is a subset of $q(\mathfrak{N})$, a convex non-definable set without minimal and maximal elements. Therefore such definable set is a subset of $V_q(\bar{\gamma})$ (*neighborhood of* $\bar{\gamma}$ in q). This explains the equality of two convex sets.

(P_3) If $q \in S_1(A)$ is non uniquely realizable and irrational, then for any $\bar{\gamma} \in N$, whenever $QV_q(\bar{\gamma}) \neq \emptyset$ the 1-types $q(x) \cup \{x < QV_q(\bar{\gamma})\}$ and $q(x) \cup \{QV_q(\bar{\gamma}) < x\}$ are non uniquely realizable and irrational.

By theorem of compactness and (P_2) there exists $\delta_1, \delta_2 \in q(\mathfrak{N})$ such that $\delta_1 < V_q(\bar{\gamma}) < \delta_2$ and since q is non uniquely realizable that is there exists definable over A monotonic bijection $f: q(\mathfrak{N}) \to q(\mathfrak{N})$, neighbourhood $V_q(\mathfrak{N})$ can not have minimal and maximal element. Concerning that for irrational type q, $q(\mathfrak{N})$ is a convex non-definable set without maximal and minimal elements, $r_1 := tp(\delta_1 | A\bar{\gamma})$ and $r_2 = tp(\delta_2 | A\bar{\gamma})$ are irrational and because $f(V_q(\bar{\gamma})) = V_q(\bar{\gamma})$, where f acts on $r_1(\mathfrak{N})$ and $r_2(\mathfrak{N})$. This means r_1 and r_2 are non uniquely realizable.

Denote $p_n(x) := p(x) \cup (QV_p(\bar{\alpha}_{n-1}) < x)$. Then by (P_2), (P_3) p_n is non uniquely realizable, irrational and finitely satisfiable in M because the right sides of p and p_n coincide

For any $n, k, m < \omega$,

$$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) < QV_{p_n}(\alpha_{n+k+1}, \alpha_{n+k+2}, \dots, \alpha_{n+k+m}), \qquad (1)$$

and therefore, all these sets have empty intersection.

Proof of (1) is done by induction on m. Assume (1) for m.

Let $r_1(x) = p_n \cup (x < QV_{p_n}(x)(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}))$ and

$$r_2(y) = p_n(y) \cup (QV_{p_n}(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}) < y).$$

Suppose that

$$QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \cap QV_{p_n}(\alpha_{n+k+1}, \dots, \alpha_{n+k+m}, \alpha_{n+k+m+1}) \neq \emptyset.$$

As long as the first set does not change, there exists a formula $L(x, \alpha_{n+k+m+1})$ definable over $M\bar{\alpha}_n\alpha_{n+k+1} \dots \alpha_{n+k+m+1}$ such that

$$L(\mathfrak{N}, \alpha_{n+k+m+1}) \subset QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r(\mathfrak{N}).$$

Let β be the endpoint of one of the intervals of formula L, then since p_n is non uniquely realizable,

$$\beta \in QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r(\mathfrak{N}).$$

As long as $\beta \vDash r_1$ and $\alpha_{n+k+m+1} \vDash r_2$ by (P_1) there exists monotonic function $f: r_2(\mathfrak{N}) \to r_1(\mathfrak{N})$ definable over $M\bar{\alpha}_n\alpha_{n+k+1} \dots, \alpha_{n+k+m}$ such that $f(\alpha_{n+k+m+1}) = \beta$. However $\beta \in QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k})$ and therefore, there is 1-formula $H(x)$ over $M\bar{\alpha}_{n+k}$ such that

$$\beta \in H(\mathfrak{N}) \subset QV_{p_n}(\alpha_n, \alpha_{n+1}, \dots, \alpha_{n+k}) \subset r_1(\mathfrak{N}).$$

Then $\alpha_{n+k+m+1}$ belongs to set definable over $M\bar{\alpha}_{n+k+m}$ $f^{-1}(H(\mathfrak{N})) \subset r_2(\mathfrak{N})$. This means $\alpha_{n+k+m+1} \in QV_p(\bar{\alpha}_{n+k+m})$. Contradiction.

From (P_1) and (1) it follows that for every $r \in S_1(M)$, whenever for any $i < n$, $p_i \perp^w r$ and $p_n \not\perp^w r$

$$QV_r(\bar{\alpha}) \cap QV_r(\bar{\alpha}_i, \alpha_{n+i+1}, \dots, \alpha_{2n+1}) = \emptyset \qquad (2)$$

Let for Σ^+-formula $\psi(x, \bar{y})$ corresponding formula of signature Σ is $K_{\psi(x,\bar{y})}(x, \bar{y}, \bar{\alpha}_n)$. Therefore to have the solution in M for any formula $K_\psi(x, \bar{a}, \bar{\alpha}_n)$ it is sufficient to write the formula

$$K_{\exists x\psi(x,\bar{y})}(\bar{y}, \bar{\alpha}_{2n+1}) := \exists x(K_\psi(x, \bar{y}, \bar{\alpha}_n) \wedge$$
$$\wedge_{i \le n} K_\psi(x, \bar{y}, \bar{\alpha}_{n-i}, \alpha_{(n-i)+n+1}, \alpha_{(n-i)+n+2} \dots, \alpha_{2n+1})).$$

In 1996 B.S. Baizhanov dealed with problem on model of weakly ordered minimal theory concerning the unary convex predicate expantion of a model. He submitted in the Journal of Symbolic Logic his results of obtaining a systematization of 1-types over a subset of a model of weakly ordered minimal theory. His arlicle was published in 2001 [8, P. 1382]. Ye.Baisalov and B. Poizat studied "beautiful" pairs of ordered minimal theories models. In 1996 they verified the $\exists x \in M$ quantifier elimination [90]. Hard to say that are the approaches in [90, P. 570] and [44, P. 3] are different, because they have similar principles (i)-(ii) from [44, P. 3].

We say that the expansion $\mathfrak{M}^+$ by all externally definable subsets admits quantifier elimination, whenever for every formula $\phi(\bar{y})$ of signature Σ^+ there exists a formula of signature $K_\phi(\bar{y}, \bar{z})$ Σ, and the element $\bar{\alpha} \in N \backslash M$ that for every $\bar{a} \in M$ the proceeding is true:

$$\mathfrak{M}^+ \vDash \phi(\bar{a}) \Longleftrightarrow \vDash K_\phi(\bar{a}, \bar{\alpha}).$$

Approach of Shelah. In 2004 [91] a model of NIP theory was examined and quantifier elimination holds true for the expansion by all externally definable subsets that means that it is NIP was verified by S. Shelah. The biggest issue is the "there exists in the submodel" quantifier elimination. Towards a contradiction he proposed an indiscernible sequence $\langle \bar{b}_n : n < \omega \rangle$ and shown that "there exists x in the submodel" quantifier elimination for $\varphi(x, \bar{a})$ failure implies that $\varphi(\alpha, \bar{b}_n)$ is true whenever for some α n is even from that it can be proven that the theory has IP.

In 2005 Shelah's simplified proof was found by V.V. Verbovskiy and F. Wagner [92], in short by using the notion of a finitely realizable type. In 2006 Shelah's theorem additional two re-proofs were given by A. Pillay using the notions of quantifier-free heirs and types and for the other re-proof he used the notions of quantifier-free coheirs and types.

The analysis of approaches shows that we can control the number of one-types realizations using the theory of orthogonality [93]. The generalization of notions of neighborhood and quasi-neighborhood it is possible to formulate the following theorem

Theorem 5.2.1 Consider a complete theory T *that for any set A the next holds:*

1) $\forall$ type $p\ from\ S_1(A)$, for each tuple $\bar{\gamma}$, that $QV_p(\bar{\gamma}) = V_p(\bar{\gamma})$

2) $\forall$ type $p, q \in S_1(A)$ the proceeding is true. If $p \not\perp^a q$, then $q \not\perp^a p$.

Then the expansion by all externally definable subsets of model of the theory T admits the elimination of quantifiers.

Theorem 5.2.2 [94] *Let M be a weakly o-minimal ordered group of signature Σ, N be an elementary extension of M, i.e. $N \succ M$. Suppose that $\alpha \in N \backslash M$ such that $p = tp(\alpha|M)$ is irrational, and $M_\alpha^+ := \langle M; \Sigma^+ \rangle$ is an expansion of M by U^2, such that it admits a uniform representation of Σ^+-formulas by Σ-formulas. Then M^+ preserves both weak o-minimality and group properties.*

Proof: Consider an expansion of a weakly o-minimal group $M = \langle Q; =, <, + \rangle$ by predicate $U(x, y)$, such that

$$M_\alpha^+ \vDash U(a, b) \Leftrightarrow N \vDash a < b < a + \alpha.$$

Let's show that this expansion is externally definable.

Consider an arbitrary quantifier free formula $H^+(x_1, \dots, x_n)$ of signature Σ^+.

This formula can be represented in disjunctive normal form:

$$H^+(x_1,\dots,x_n) = \vee\ (\wedge\ H_k(x_1,\dots,x_n) \wedge \wedge\ U(x_i,x_j)),$$

where $i,j \in \{1,\dots,n\}$ and H_k are atoms in language Σ. Then we can swap every $U(x_i,x_j)$ with $x_i < x_j \wedge x_j < x_i + \alpha$, thus we get quantifier free formula $K_{H^+}(x_1,\dots,x_n,\alpha)$ such that

$$\forall a_1,\dots,a_n\ \ M_\alpha^+ \vDash H^+(a_1,\dots,a_n) \Leftrightarrow N \vDash K_{H^+}(a_1,\dots,a_n,\alpha)$$

Continue by induction. Let

$$\forall a \forall \bar{b}\ \ M_\alpha^+ \vDash \Phi^+(a,\bar{b}) \Leftrightarrow N \vDash K_{\Phi^+}(a,\bar{b},\alpha)$$

holds for any formula $\Phi^+(x,\bar{y})$ of depth n. Consider a formula $\exists x \Phi^+(x,\bar{b})$ of depth $n+1$. We should find a formula $K_{\exists x \Phi^+}(x,\bar{b})$, such that $K_{\Phi^+}(N,\bar{b},\bar{\alpha}) \cap M \neq \emptyset$ whenever $M_\alpha^+ \vDash \exists x \Phi^+(x,\bar{b})$. By theorem 58 from [8, P. 1382] $\forall \bar{\alpha} \in N \backslash M\ \ \exists \bar{\beta} \in N$, such that for any formula $K(x,\bar{b},\bar{\alpha})$ there exists a formula $K'(\bar{y},\bar{\alpha},\bar{\beta})$ such that

$$\forall \bar{b} \in M[K(N,\bar{b},\bar{\alpha}) \cap M \neq \emptyset \Leftrightarrow N \vDash K'(\bar{b},\bar{\alpha},\bar{\beta})]$$

Consider a formula $K_{\Phi^+}(x,\bar{b},\bar{\alpha})$, the set $K_{\Phi^+}(N,\bar{b},\bar{\alpha})$ is a definable in weakly o-minimal group structure. Thus it is union of finite number of convex sets. $K_{\Phi^+}(N,\bar{b},\bar{\alpha}) \cap M$ does not increase (in case some convex sets are cover irrational cut, two convex sets stick into one convex set). Thus every single definable set in M_α^+ is union of finite number of convex sets, and it follows by the definition that it is weakly o-minimal.

CONCLUSION

The dissertation considers expansions of models of NIP theories. Such theories include the following: linearly ordered theories, countably categorical theories, weakly o-minimal theories, theories of finite convexity rank. The aim was to investigate preservation of certain properties(countable categoricity, weak o-minimality, convexity rank) of models by expanding by unary predicates, equivalence relations or binary predicates. The key results in this context are as follows:

Expansion of model of countably categorical weakly o-minimal theory of finite convexity rank by a finite family of convex unary predicates preserves countable categoricity and convexity rank. Similar result for quite o-minimal countably categorical theories: Expansion of model of countably categorical quite o-minimal theory of finite convexity rank by a finite family of convex unary predicates preserves countable categoricity and convexity rank.

More complicated result for expansion by equivalence relations: Touchstone for maintaining both countable categoricity and weak o-minimality (and in addition to this the 1-indiscernibility) when expanding a model of a 1-indiscernible countably categorical weakly o-minimal theory of finite convexity rank by an equivalence relation partitioning the universe into infinitely many infinite convex classes.

Results on expansions by binary predicates: Touchstone for maintaining countable categoricity for a 1-indiscernible weakly o-minimal expansion of countably categorical weakly o-minimal theory of convexity rank 1 by every single binary predicate.

Touchstone for maintaining countable categoricity for a weakly o-minimal expansion of a non-1-indiscernible countably categorical weakly o-minimal theory of convexity rank 1 by random binary predicate.

Maintaining weak o-minimality when expanding a weakly o-minimal ordered group by an externally definable binary predicate.

Assessment of the completeness of the aims of the work. The results of investigation are new received using on our own tools and methods. Conditions of preservation of either weak o-minimality, countable categoricity under expansion by unary or binary predicates were found. Consequently, the goals of the work have been entirely accomplished.

Suggestions on applications of the obtained results. Obtained results in this field of model theory can be used throughout the study of models of NIP theories, particularly expansions of weakly o-minimal theories. Results on the expansions by externally definable sets can be applied to theories of algebraic structures.

Assessment of scientific level of the work in comparison with the achievements in the scientific direction. The findings obtained in accordance with the best contributions of foreign colleagues are not lacking and add to the study of the expansion of models of NIP theories..

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